CHAPTER 4

INFLATIONARY MODEL BUILDING

Essentially, all models are
wrong, but some are useful.

George E. P. Box, 1987

As we learnt in the previous chapter, inflation is not a model, but rather a paradigm including hundreds of particular models parametrizing a very simple idea: the shrinking of the Hubble radius in the early Universe. Different inflationary models are characterized by the particular choice of new physics assumed to produce this effect. Whatever the mechanism, it must be equipped with a *graceful exit* able to end the accelerated expansion. In this chapter we discuss the simplest possibility: a canonical scalar field minimally coupled to gravity.

4.1 Canonical scalar field dynamics

Consider the action

$$S = \int \mathrm{d}^4x \sqrt{-g} \left[\frac{1}{2\kappa^2} R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi) \right] , \tag{4.1}$$

with $\kappa \equiv M_P^{-1}$ the inverse of the reduced Planck mass. The first term is the usual Einstein-Hilbert action. The last two pieces are the kinetic and potential contributions of a canonically-normalized scalar field minimally coupled to gravity, the *inflaton* $\phi = \phi(t, \mathbf{x})$.

Non-minimal coupling and $f(R)$ theories

The analysis presented in this section can be easily extended to theories containing non-minimal couplings to gravity $\sim f(\phi)R$ or $f(R)$ modified gravity theories. When written in the Einstein frame all these theories can be reduced to the form (4.1).

The energy-momentum tensor for the inflation field is obtained by varying the action (4.1) with respect to the metric

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} \partial^\sigma \phi \partial_\sigma \phi + V \right) . \tag{4.2}$$

The variation with respect to ϕ gives us the Klein-Gordon equation of motion

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\partial^\mu\phi)+V_{,\phi}=0\,. \tag{4.3}$$

Exercise
Derive Eqs. (4.2) and (4.3).

For a *background* FLRW geometry, the evolution of the inflaton field can only depend on time. In this limit, the Klein-Gordon equation (4.1) becomes

$$\ddot{\phi}+3H\dot{\phi}+V_{,\phi}=0\,, \tag{4.4}$$

with $V_{,\phi}=dV/d\phi$ the first derivative of the inflationary potential. Note that the term $3H\dot{\phi}$ in this expression is a friction term which tends to decrease the velocity of the field. The strength of the friction is determined by the Friedmann equations

$$H^2=\frac{\kappa^2}{3}\rho_\phi\,, \qquad \dot{H}+H^2=-\frac{\kappa^2}{6}(\rho_\phi+3p_\phi)\,, \tag{4.5}$$

with ρ_ϕ and p_ϕ the energy density and pressure of the scalar field

$$\rho_\phi=\frac{1}{2}\dot{\phi}^2+V\,, \qquad p_\phi=\frac{1}{2}\dot{\phi}^2-V\,. \tag{4.6}$$

Exercise
Show that Eqs. (4.4) and (4.5) are not independent.

Combining the expressions in (4.5) we can rewrite the Hubble flow parameter (3.48) as

$$\epsilon\equiv-\frac{\dot{H}}{H^2}=\frac{3}{2}(1+w_\phi)=\frac{\kappa^2}{2}\frac{\dot{\phi}^2}{H^2}\,, \tag{4.7}$$

with

$$w_\phi\equiv\frac{p_\phi}{\rho_\phi}=\frac{\frac{1}{2}\dot{\phi}^2-V}{\frac{1}{2}\dot{\phi}^2+V} \tag{4.8}$$

the scalar field equation-of-state. Note that w_ϕ is generically time-dependent, since the energy density ρ_ϕ can be arbitrarily distributed among kinetic and potential contributions.

4.1.1 Slow-roll approximation

A simple inspection of Eq. (4.7) reveals that inflation can take place for $w_\phi<-1/3$. In what follows we will focus on the quasi de Sitter limit $w_\phi\approx-1$.[1] In this limit, the potential energy of the scalar field dominates over its kinetic counterpart,

$$\dot{\phi}^2\ll V\,, \tag{4.9}$$

[1]Note that this is the minimal value of the equation of state parameter for any positive potential V. Although the strong energy condition $\rho+3p>0$ is violated, the weak energy condition $\rho+p\geq 0$ is still satisfied. A fluid with $p<-\rho$ is said to be "phantom".

and the inflaton field *rolls slowly*. If the potential is sufficiently flat, the condition (4.9) will be eventually satisfied due to the friction term in Eq. (4.4) (we will come back to this point in Section 4.1.2). When that happens,

$$w_\phi \approx -1 \qquad \Rightarrow \qquad \rho_\phi \simeq \text{const.} \qquad \Rightarrow \qquad H \simeq \text{const.} \qquad \Rightarrow \qquad a(t) \sim e^{Ht}\,. \tag{4.10}$$

Dilution of pre-inflationary particle content

Note that while the energy density of the inflaton field remains approximately constant during inflation, the energy density of any other matter or radiation content existing at the onset of the inflationary regime becomes exponentially suppressed

$$\rho_M \propto a^{-3} \sim e^{-3Ht}\,, \qquad\qquad \rho_R \propto a^{-4} \sim e^{-4Ht}\,. \tag{4.11}$$

At the end of inflation, most of the energy of the Universe is stored in the zero mode of the inflaton field. In order for the thermal history of the Universe to start, the energy sitting in the inflaton condensate must be transferred to the SM particles in a direct or indirect way. This relocation of energy is called *reheating*.

In order to sustain the accelerated expansion for a sufficient number of e-folds, the inflaton acceleration must be small

$$|\ddot\phi| \ll |3H\dot\phi|, |V_{,\phi}|\,. \tag{4.12}$$

Under these conditions Eqs. (4.4) and (4.5) become

$$H^2 \simeq \frac{\kappa^2}{3} V \simeq \text{constant}\,, \tag{4.13}$$

$$3H\dot\phi \simeq -V_{,\phi}\,. \tag{4.14}$$

In the slow-roll approximation, the inflationary conditions (3.48) and (3.50) become simple restrictions on the shape of the inflationary potential. To see this, consider the derivative of Eqs. (4.7) and (4.14) with respect to time

$$\dot\epsilon = \kappa^2 \left(\frac{\dot\phi\ddot\phi}{H^2} - \frac{\dot\phi^2 \dot H}{H^3} \right)\,, \qquad\qquad 3\dot H\dot\phi + 3H\ddot\phi \simeq -V_{,\phi\phi}\dot\phi\,, \tag{4.15}$$

Taking into account these expressions together with Eqs. (4.13) and (4.14), we can rewrite Eqs. (3.48) and (3.50) as[2]

$$\epsilon \simeq \epsilon_V\,, \qquad\qquad \eta \simeq 4\epsilon_V - 2\eta_V\,, \tag{4.16}$$

with[3]

$$\epsilon_{\rm V} \equiv \frac{1}{2\kappa^2}\left(\frac{V_{,\phi}}{V}\right)^2\,, \qquad\qquad \eta_{\rm V} \equiv \frac{1}{\kappa^2}\frac{V_{,\phi\phi}}{V}\,, \tag{4.17}$$

[2] When the slow-roll conditions are not satisfied the relation between (ϵ, η) and (ϵ_V, η_V) is not linear.
[3] Note that while ϵ_V is positive definite, η_V can be either positive of negative.

the so-called *slow-roll parameters*. The inflationary conditions $\epsilon, |\eta| < 1$ translate therefore into the requirements

$$\epsilon_{\rm V}, |\eta_{\rm V}| \ll 1\,. \tag{4.18}$$

Exercise
Show this.

Inflation ends when the slow-roll parameter equals one

$$\epsilon(\phi_{\rm end}) \simeq \epsilon_{\rm V}(\phi_{\rm end}) = 1\,. \tag{4.19}$$

As explained in Section 3.3.1, the duration of inflation is usually expressed in terms of the *number of e-folds* of accelerated expansion,

$$N = \int_{a_i}^{a_{\rm end}} d\ln a\,, \tag{4.20}$$

with $a_{\rm end}$ the scale factor at the end of inflation and $a_i < a_{\rm end}$ a fiducial value before that point. With this definition, the number of e-folds decreases during the inflationary phase and reaches zero at its end. Using Eq. (4.7), the number of e-folds (4.20) can be alternatively written as

$$N = \int_{t_i}^{t_{\rm end}} H{\rm d}t = \int_{\phi_i}^{\phi_{\rm end}} \frac{H}{\dot\phi}{\rm d}\phi = \int_{\phi_i}^{\phi_{\rm end}} \frac{\kappa\,|{\rm d}\phi|}{\sqrt{2\epsilon(\phi)}} \simeq \int_{\phi_i}^{\phi_{\rm end}} \frac{\kappa\,|{\rm d}\phi|}{\sqrt{2\epsilon_{\rm V}(\phi)}}\,, \tag{4.21}$$

with $\phi_{\rm end}$ given by Eq. (4.19). This expression allows to compute the fiducial field value ϕ_i as a function of the numbers e-folds N. If the slow-roll parameter ϵ_V were approximately constant during inflation, the change in the field would be of order

$$\kappa|\Delta\phi| \approx \sqrt{2\epsilon}N\,. \tag{4.22}$$

Note that for $N = 60$, this translates into a trans-Planckian excursion $\kappa|\Delta\phi| \sim O(1)$, unless $\epsilon \lesssim 10^{-4}$. We will come back to this point in Section 4.2.

Exercise
Compute the slow-roll parameters $\epsilon_{\rm V}$ and $\eta_{\rm V}$ as a function of the number of e-folds N for the inflationary potential

$$V = \frac{\lambda}{4}\phi^4\,. \tag{4.23}$$

4.1.2 Attractor solution

The conditions (4.18) are *necessary* but not *sufficient* conditions. The second-order character of Eq. (4.4) allows to choose the initial value of the field velocity $\dot\phi$ in such a way that the slow-roll approximation is violated. Note however that this equation coincides with the equation of motion a particle rolling down a potential V under the influence of a friction term $3H\dot{\phi}$. Like for a particle trajectory, one should expect an attractor solution for large enough friction. To derive this attractor solution let us introduce the so-called *Hamilton-Jacobi* formulation.

In the Hamilton-Jacobi formulation the scalar field itself is taken to be the time variable. Differentiating the Friedmann equation (4.5) with respect to time and using the Klein-Gordon equation (4.4) we obtain

$$2\dot{H} = -\kappa^2\dot{\phi}^2\,, \qquad \Rightarrow \qquad \dot{\phi} = -\frac{2}{\kappa^2}H_{,\phi}\,, \tag{4.24}$$

where in the last step we have assumed $\dot{\phi}$ to be a monotonic function of time.[4] Using (4.24), the Friedmann equation (4.5) can be recast into the first-order form

$$\frac{2}{\kappa^2}H_{,\phi}^2 - 3H^2 = -\kappa^2 V\,. \tag{4.25}$$

This equation is the so-called *Hamilton-Jacobi equation*. The existence of an attractor solution in Eq. (4.25) can be shown by considering the influence of a linear homogeneous perturbation $\delta H(\phi)$ on an exact solution $H_0(\phi)$. Substituting

$$H(\phi) = H_0(\phi) + \delta H(\phi) \tag{4.26}$$

into Eq. (4.25) and linearizing the result, we get a differential equation for the linear perturbation $\delta H(\phi)$, namely

$$H_{0,\phi}\delta H_{,\phi} \simeq \frac{3\kappa^2}{2}H_0\delta H\,. \tag{4.27}$$

The general solution of this equation can be written as

$$\delta H(\phi) = \delta H(\phi_i)\exp\left(\frac{3\kappa^2}{2}\int_{\phi_i}^{\phi}\frac{H_0(\phi)}{H_{0,\phi}(\phi)}d\phi\right)\,, \tag{4.28}$$

with $\delta H(\phi_i)$ the value of the perturbation at some initial value ϕ_i and $\Delta N > 0$ the number of e-folds. The linear perturbations around a given solution, inflationary or not, die away as the scalar field evolves. The decay of perturbations is particularly dramatic for inflationary solutions ($H_0 \simeq const.$)

$$\delta H(\phi) = \delta H(\phi_i)\exp\left[-3\Delta N\right]\,. \tag{4.29}$$

4.2 Slow-roll inflation and EFT

In an effective field theory framework, the simple Lagrangian density (4.1) must be complemented by an infinite series of higher dimensional operators[5]

$$S_{\rm EFT}[\phi] = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi) + \sum_i \frac{c_i\,\mathcal{O}_i[\phi]}{\Lambda^{\delta_i-4}}\right]\,, \tag{4.30}$$

with Λ a cutoff scale that is assumed to exceed the Hubble scale during inflation ($\Lambda \gtrsim H$) and order 1 coefficients c_i. As we explain now, these higher dimension operators could threaten the flatness of the original potential $V(\phi)$.

[4] For definiteness we take $\dot{\phi} > 0$.
[5] For simplicity we stay at the lowest order in derivative interactions.

4.2.1 UV sensitivity

Let us assume for concreteness that $V(\phi)$ is renormalizable and contains operators up to dimension 4. Denoting by V_0 the typical scale of the potential $V(\phi)$ during inflation, the higher dimensional operators in Eq. (4.30) give rise to corrections of order

$$\mathcal{O}_{\delta_i} = c_i V_0 \left(\frac{\phi}{\Lambda}\right)^{\delta_i - 4} . \tag{4.31}$$

What value should we choose for the cutoff Λ? We don't really know. The precise value of Λ depends on the different thresholds (masses of new particles) that were integrated out to get the low-energy effective field theory. In principle, it could be as large as the Planck mass M_P, where gravitational interactions become important for sure. Naively, one would tend to believe that these highly suppressed operators do not have any impact on physics happening at the typical energy of inflation. However these higher dimensional operators could threaten the flatness of the original potential $V(\phi)$. Whenever the scalar field ϕ traverses a distance $\Delta\phi \gg M_P$, the effective action receives substantial corrections from the infinite set of higher-dimension operators. In order for the resulting effective potential to support inflation, it should remain sufficiently flat. This requirement translates into an unnatural finetuning of the infinitely many coefficients c_i or into the existence of additional symmetries in the UV theory (such as shift symmetry or supersymmetry) able to forbid the generation of higher dimensional operators. Why the inflaton couples so weakly to Planck-scale degrees of freedom it still an open question.

Still far from the quantum gravity regime

The super-Planckian displacements of the inflaton field *do not* lead to Planckian energy densities since, as we will see in the following chapters, the normalization of scalar fluctuations leading to temperature fluctuations in the CMB requires that $V \ll M_P^4$. For the observable window of inflationary e-folds the semiclassical description of gravity is always accurate.

4.2.2 Shift symmetry from scale invariance

To illustrate how particular symmetries can control the UV sensitivity of inflationary observables let us consider a single real scalar field non-minimally coupled to gravity with action

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2 + \xi\phi^2}{2}R - \frac{(\partial\phi)^2}{2} - \frac{\lambda\phi^4}{4}\right) , \tag{4.32}$$

and $\xi \gg 1$. The non-minimal coupling to curvature modifies the scalar field kinetic term at large field values, leading to slow-roll inflation even if the inflationary potential $\lambda\phi^4/4$ is not sufficiently flat. Note the emergent symmetry of Eq. (4.32) at $\phi \gg M_P/\sqrt{\xi}$. In this limit the action does not contain dimensionfull parameters and becomes approximately invariant under the simultaneous (scale) transformations

$$x^\mu \to \alpha^{-1}x^\mu \qquad g_{\mu\nu}(x) \to g_{\mu\nu}(\alpha^{-1}x) , \qquad \phi(x) \to \alpha\phi(\alpha^{-1}x) ,$$

with α a constant.

Due to non-minimal coupling to gravity the scalar sector in Eq. (4.32) becomes non-renormalizable, meaning that the theory should be interpreted as an EFT valid till a given cutoff scale Λ. Although quite natural, the identification of the cutoff scale with the Planck mass M_P may turn out to be theoretically inconsistent since other processes can break tree-level unitarity at lower energies. To determine the true cutoff scale of the theory we expand the scalar field and the metric around their background values,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,, \qquad\qquad \phi = \bar{\phi} + \delta\phi\,, \tag{4.33}$$

and look for the coefficients of higher dimensional operators. Let us start by computing the quadratic Lagrangian for perturbations. We get

$$\begin{aligned}\mathcal{L}^{(2)} = {} & \frac{M_P^2 + \xi\bar{\phi}^2}{8}\left(h^{\mu\nu}\Box h_{\mu\nu} + 2\partial_\nu h^{\mu\nu}\partial^\rho h_{\mu\rho} - 2\partial_\nu h^{\mu\nu}\partial_\mu h - h\Box h\right) \\ & - \frac{1}{2}(\partial_\mu\delta\phi)^2 + \xi\bar{\phi}\left(\Box h - \partial_\lambda\partial_\rho h^{\lambda\rho}\right)\delta\phi\,,\end{aligned} \tag{4.34}$$

with $h = h^\mu_\mu$ denoting the trace of the excitations. The last term in Eq. (4.34) mixes the trace of the metric with the scalar perturbation $\delta\phi$. In order to diagonalize the kinetic terms we must perform a field perturbation redefinition

$$\delta\phi = \sqrt{\frac{M_P^2 + \xi\bar{\phi}^2}{M_P^2 + \xi\bar{\phi}^2 + 6\xi^2\bar{\phi}^2}}\,\delta\hat{\phi}\,, \tag{4.35}$$

$$h_{\mu\nu} = \frac{1}{\sqrt{M_P^2 + \xi\bar{\phi}^2}}\,\hat{h}_{\mu\nu} - \frac{2\xi\bar{\phi}\bar{g}_{\mu\nu}}{\sqrt{(M_P^2 + \xi\bar{\phi}^2)(M_P^2 + \xi\bar{\phi}^2 + 6\xi^2\bar{\phi}^2)}}\,\delta\hat{\phi}\,. \tag{4.36}$$

Exercise
Derive Eq. (4.34) and show that it can be reduced to a diagonal form via the change of variables in Eqs. (4.35) and (4.36).

Once the kinetic sector has been reduced to a diagonal form we can proceed to read the cutoff scale from the operators with dimension higher than four. The leading order operator is the cubic scalar–graviton interaction. In terms of the canonically normalized perturbations $(\delta\hat{\phi}, \hat{h})$ it takes the form,

$$\frac{\xi\sqrt{M_P^2 + \xi\bar{\phi}^2}}{M_P^2 + \xi\bar{\phi}^2 + 6\xi^2\bar{\phi}^2}(\delta\hat{\phi})^2\Box\hat{h}\,. \tag{4.37}$$

As usual, the cutoff is identified with the inverse of the coefficient of this higher dimensional operator, namely

$$\Lambda(\bar{\phi}) = \frac{M_P^2 + \xi\bar{\phi}^2 + 6\xi^2\bar{\phi}^2}{\xi\sqrt{M_P^2 + \xi\bar{\phi}^2}}\,. \tag{4.38}$$

Note that Λ is not just a number but rather depends on the expectation value of the background field ϕ. At small field values ($\bar{\phi} \lesssim M_P/\xi$), the cutoff is approximately constant

$$\Lambda(\bar{\phi}) \simeq \frac{M_P}{\xi}\,, \tag{4.39}$$

while for $M_P/\xi \ll \bar{\phi} \ll M_P/\sqrt{\xi}$ we rather have a field-dependent cutoff

$$\Lambda(\bar{\phi}) \simeq \frac{\xi\bar{\phi}^2}{M_P}\,, \tag{4.40}$$

which is still below the Planck mass M_P but starts to grow. At $\bar{\phi} \gg M_P/\sqrt{\xi}$ the suppression becomes strong,

$$\Lambda(\bar{\phi}) \simeq \sqrt{\xi}\bar{\phi}\,, \tag{4.41}$$

and coincides with the *dynamical* Planck mass in that limit, $M_P^{\rm eff} = \sqrt{M_P^2 + \xi\bar{\phi}^2} \simeq \sqrt{\xi}\bar{\phi}$.

A sensible computation of radiative corrections within the EFT framework requires to complement the tree-level action (4.32) with an infinite number of higher-dimensional operators suppressed by the cutoff scale (4.38), namely

$$S = \int d^4x\sqrt{-g}\left(\frac{M_P^2 + \xi\phi^2}{2}R - \frac{(\partial\phi)^2}{2} - V_{\rm EFT}(\phi)\right)\,, \tag{4.42}$$

with

$$V_{\rm EFT}(\phi) = \frac{\lambda\phi^4}{4} + \sum_i \frac{c_i\,\mathcal{O}_i[\phi]}{\Lambda(\phi)^{\delta_i-4}} = \frac{\lambda\phi^4}{4}\left(1 + \frac{1}{\phi^4}\sum_i \frac{\bar{c}_i\,\mathcal{O}_i[\phi]}{\Lambda(\phi)^{\delta_i-4}}\right)\,, \tag{4.43}$$

and $\bar{c}_i \equiv 4c_i/\lambda$. Note that all the higher dimensional operators become scale-invariant in the large field regime $\phi \gg M_P/\sqrt{\xi}$. The impact of these corrections on the inflationary dynamics can be more easily understood in a frame in which the gravitational part of the action takes the usual Einstein-Hilbert form. This frame, usually called the *Einstein frame*, is obtained by performing a *conformal transformation*

$$g_{\mu\nu} = \Omega^{-2}(x)\tilde{g}_{\mu\nu}\,, \qquad g^{\mu\nu} = \Omega^2(x)\tilde{g}^{\mu\nu}\,, \qquad \sqrt{-g} = \Omega^{-4}(x)\sqrt{-\tilde{g}}\,, \tag{4.44}$$

with $\Omega(x)$ an appropriate function called the *conformal factor*. Taking into account that under the metric redefinition (4.44) the scalar curvature transforms as

$$R = \Omega^2(x)\left(\tilde{R} + 3\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu \ln\Omega^2(x) - \frac{3}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \ln\Omega^2(x)\tilde{\nabla}_\nu \ln\Omega^2(x)\right)\,, \tag{4.45}$$

we can rewrite Eq. (4.42) as

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{M_P^2 + \xi\phi^2}{2\Omega^2(x)}\left(\tilde{R} + 3\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu \ln\Omega^2(x) - \frac{3}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu \ln\Omega^2(x)\tilde{\nabla}_\nu \ln\Omega^2(x)\right)\right.$$
$$\left.-\frac{1}{2\Omega^2(x)}\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tilde{V}_{EFT}(\phi)\right]\,, \tag{4.46}$$

with

$$\tilde{V}_{EFT}(\phi) \equiv \frac{V_{EFT}(\phi)}{\Omega(x)^4}\,. \tag{4.47}$$

A simple inspection of Eq. (4.46) reveals that in order to recover the usual Einstein-Hilbert term we must choose a conformal factor

$$\Omega^2(x) = \frac{M_P^2 + \xi\phi^2}{M_P^2} \,. \tag{4.48}$$

Taking into account this equation and integrating by parts the second term in Eq. (4.46) we get

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} h(\phi)^2 \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \tilde{V}_{EFT}(\phi) \right] \,,$$

with

$$h(\phi)^2 = \frac{M_P^2 (M_P^2 + (1+6\xi)\xi\phi^2)}{(M_P^2 + \xi\phi^2)^2} \,. \tag{4.49}$$

In the new frame the field ϕ is no longer coupled to the scalar curvature and the gravitational sector is described by the usual Einstein-Hilbert term. The price to pay is that the part of the action that describes the dynamics of the scalar field is modified in a non-trivial way, in particular it contains a non-canonical kinetic term. In order to canonically normalized the kinetic we make use of the chain rule for the derivatives of ϕ to obtain

$$h(\phi)^2 \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = h(\phi)^2 \left(\frac{d\phi}{d\chi} \right)^2 \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi \,, \tag{4.50}$$

with $\chi \equiv \chi(\phi)$. Requiring $h(\phi) \frac{d\phi}{d\chi} = 1$ we get

$$\frac{d\chi}{d\phi} = \frac{M_P \sqrt{M_P^2 + (1+6\xi)\xi\phi^2}}{M_P^2 + \xi\phi^2} \simeq \frac{M_P \sqrt{M_P^2 + 6\xi^2\phi^2}}{M_P^2 + \xi\phi^2} \,, \tag{4.51}$$

where in the last step we have made use of the assumption $\xi \gg 1$. In the limit $\xi\phi \gg M_P$, the differential equation (4.51) can be easily integrated, to obtain

$$\frac{d\chi}{d\phi} \simeq \frac{\sqrt{6} M_P \xi \phi}{M_P^2 + \xi\phi^2} \quad \to \quad \chi = \sqrt{\frac{3}{2}} M_P \ln \left[1 + \frac{\xi\phi^2}{M_P^2} \right] \,. \tag{4.52}$$

Solving for ϕ,

$$\phi = \frac{M_P}{\sqrt{\xi}} \left(e^{\sqrt{\frac{2}{3}} \frac{\chi}{M_P}} - 1 \right)^{1/2} \,, \tag{4.53}$$

we can rewrite the Einstein-frame action (4.49) as

$$S = \int d^4x \sqrt{-\tilde{g}} \left[\frac{M_P^2}{2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{V}_{EFT}(\phi(\chi)) \right] \,,$$

with $\tilde{V}_{EFT}(\phi(\chi))$

$$\tilde{V}_{EFT}(\phi(\chi)) = \frac{\lambda M_P^4 \phi^4}{4(M_P^2 + \xi\phi^2)^2} \left(1 + \frac{1}{\phi^4} \sum_i \frac{\bar{c}_i \, \mathcal{O}_i[\phi]}{\Lambda(\phi)^{\delta_i - 4}} \right) \tag{4.54}$$

$$= \frac{\lambda M_P^4}{4\xi^2} \left(1 - e^{-\sqrt{2/3} M_P} \right)^2 \left[1 + \Delta(\phi(\chi)) \right] \,. \tag{4.55}$$

The action (4.42) is now cast in a very intuitive form: the gravitational sector is the standard Einstein-Hilbert term, whereas the scalar sector describes a canonically normalized field with a flat exponential potential corrected by some higher dimensional operators $\Delta(\phi(\chi))$. In the inflationary regime $\phi \gg M_P/\sqrt{\xi}$ these corrections are small

$$\Delta(\phi(\chi)) = \frac{1}{\phi^4}\sum_i \frac{\bar{c}_i\,\mathcal{O}_i[\phi]}{\Lambda(\phi)^{\delta_i-4}} \simeq \frac{1}{\phi^4}\sum_i \frac{\bar{c}_i\,\mathcal{O}_i[\phi]}{(\sqrt{\xi}\phi)^{\delta_i-4}} \tag{4.56}$$

$$= \sum_i \frac{\bar{c}_i}{(\sqrt{\xi})^{\delta_i-4}}\frac{\mathcal{O}_i[\phi]}{\phi^{\delta_i}} \sim \sum_i \frac{\bar{c}_i}{(\sqrt{\xi})^{\delta_i-4}}\,. \tag{4.57}$$

and do not modify the shape of the tree level potential. The asymptotic shift-symmetry of Eq. (4.54) at $\chi \gg \sqrt{3/2}M_P$ is the Einstein-frame manifestation of the approximate scale invariance at $\phi \gg M_P/\sqrt{\xi}$ we started with in the original frame. The transition to the Einstein frame is indeed analogous to the spontaneous breaking of scale invariance. The field χ can be interpreted as the pseudo-Goldstone boson of (non-linearly realized) scale-invariance.