CHAPTER 3
THE INFLATIONARY PARADIGM

Ubi materia, ibi geometria.

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3.1 The hot Big Bang paradise

In General Relativity, the Universe as a whole becomes a dynamical entity that can be modeled and measured. The combination of Einstein's theory of gravity with the Standard Model of particle physics gives rise to the successful hot Big Bang (hBB) scenario, describing the evolution of the Universe and its matter content from the first fraction of a second till the present era. The expansion of the Universe, the relative abundance of light nuclei or the discovery of the Cosmic Microwave Background (CMB) give confidence in the basic picture, the expansion and cooling of a primordial soup. Many of the key cosmological parameters describing the Universe have been accurately determined. This has led to the establishment of a precision cosmological model known as Λ Cold Dark Matter (ΛCDM). At the same time, these parameters provide useful information for particle physics. The stringent limits on the sum of neutrino masses and on the variations of fundamental constants clearly illustrate the entanglement between cosmology and high-energy physics.

3.1.1 Homogeneity and isotropy

In this course, we will not be interested in local objects such as galaxies or stars, but rather on the dynamics of the Universe *as a whole*. In particular, we will average over local structures and assume the Universe to be described by an approximately homoheneous and isotropic "gas" of matter, whose "molecules" are, for example, galaxies. On physical grounds, homogeneity means that the physical conditions are the same at every point. Isotropy *at every point* automatically implies homogeneity.

Exercise

Convince yourself that homogeneity *does not* imply isotropy. Provide some examples.

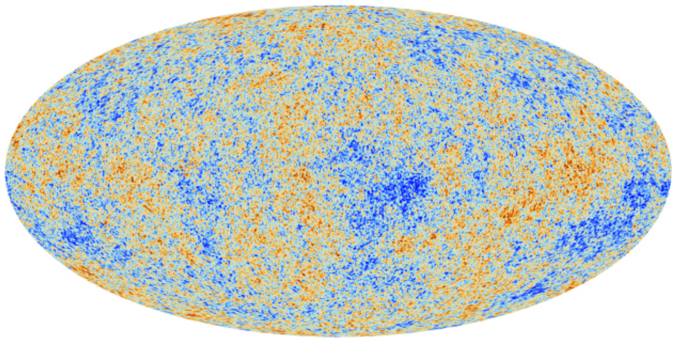

Figure 3.1: The Cosmic Microwave Background as seen by Planck. The fluctuations on top of the average temperature $T = 2.73K \simeq 0.235$ meV are one part in 10^5.

At first sight, the idealization of the Universe as an homogeneous and isotropic object might seem a bit drastic. On the other hand, we know from hydrodynamics that a continuous description of gases works very well even if these have a very discontinuous structure at molecular scales. The homogoneous and isotropic approximation seems to be indeed in good agreement with observations. Indeed. both the CMB and the galaxy distribution look rather homogeneous when averaged on sufficiently large scales (cf. Figs. 3.1 and 3.2).

A given spacetime in General Relativity is specified by its metric tensor $g_{\mu\nu}$. This quantity defines the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{3.1}$$

where dx^μ stands for infinitesimal displacements in the coordinates x^μ. From a mathematical point of view, an homogeneous and isotropic Universe must be equipped with a metric tensor invariant under translations and rotations in the spatial components. The most general 4-dimensional geometry consistent with these symmetries is the so called Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime,

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] . \tag{3.2}$$

This equation represents a time-ordered slicing of spacetime with respect to a global time t whose 3-dimensional spacial surfaces are maximally symmetric. Here r is a radial coordinate and θ and ϕ are the usual angular coordinates on a two-sphere, ranging between $0 < \theta < \pi$ and $0 \leq \phi < 2\pi$. The coordinates (r, θ, ϕ) are usually called comoving coordinates, since they are decoupled from the effect of expansion.

The FLRW metric is invariant under the redefinition

$$k \rightarrow \frac{k}{|k|} \,, \qquad r \rightarrow r\sqrt{|k|} \,, \qquad a \rightarrow \frac{a}{\sqrt{|k|}} \,, \tag{3.3}$$

meaning that the the only relevant parameter is the sign of k. We can therefore distinguish three types of spatial sections:

1. Flat: for $k = 0$ the spacial slices are flat and r ranges from zero to infinity, $0 < r < \infty$.

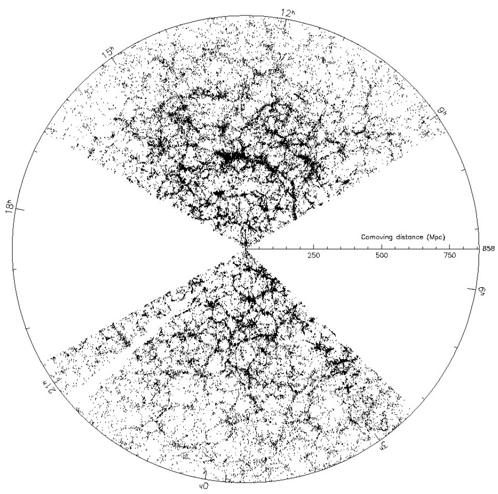

Figure 3.2: The Sloan Digital Sky Survey map. Each dot is a galaxy. The empty regions are just areas that the survey did not cover.

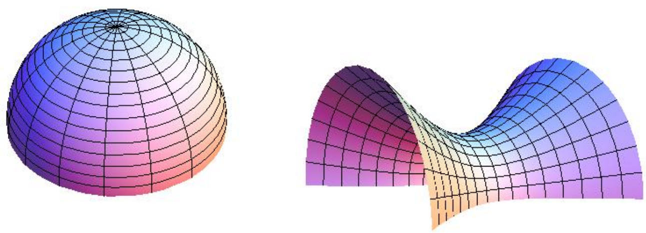

Figure 3.3: 2-dimensional projection of the 3-dimensional slices of the FLRW metric for $k = +1$ (left) and $k = -1$ (right).

2. Open: for $k = -1$ the spacial slices are hyperbolic and again $0 < r < \infty$.

3. Closed: for $k = 1$, the spacial slices are three-spheres and the radial coordinate r is restricted to a compact range, $0 < r < 1$.

The *scale factor* $a(t)$ characterizes the relative size of the spacial sections at a given time. Its temporal evolution depends on the matter content of the Universe via the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{3.4}$$

with G the Newton's constant, $R_{\mu\nu}$ the Ricci tensor, $R = g^{\mu\nu} R_{\mu\nu}$ the Ricci scalar and Λ the infamous cosmological constant. The energy-momentum tensor $T_{\mu\nu}$ encodes the Universe's matter content and is locally conserved,

$$\nabla^{\mu} T_{\mu\nu} = 0\,. \tag{3.5}$$

The homogeneity and isotropy of the background metric restricts the form of the energy-momentum tensor to the perfect fluid case

$$T_{\mu\nu} = p\, g_{\mu\nu} + (\rho + p) u_\mu u_\nu \,, \tag{3.6}$$

with u_μ the comoving four-velocity satisfying $u^\mu u_\mu = -1$ and $\rho(t)$ and $p(t)$ the local energy density and pressure of the fluid. For an observer comoving with the fluid, $u^\mu = (1, 0, 0, 0)$, the energy-momentum tensor looks isotropic

$$T^\mu{}_\nu = \mathrm{diag}(-\rho(t), p(t), p(t), p(t)) \,. \tag{3.7}$$

Note that the trace is given by

$$T = T^\mu{}_\mu = -\rho + 3p \,. \tag{3.8}$$

Exercise

Derive Eqs. (3.7) and (3.6).

3.1.2 Friedmann equations

Given the FLRW metric (3.2) metric we can compute the connection coefficients, the Ricci tensor components and the Ricci scalar. The computation of these quantities is straightforward but quite tedious, so we will simply summarize the results:

i) Connection coefficients

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1 - kr^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2 \sin^2\theta \\
&\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \\
&\Gamma^1_{22} = -r(1 - kr^2) \qquad \Gamma^1_{33} = -r(1 - kr^2)\sin^2\theta \\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
&\Gamma^2_{33} = -\sin\theta\, \cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \ .
\end{aligned} \tag{3.9}$$

ii) Ricci tensor components

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta \ ,
\end{aligned} \tag{3.10}$$

iii) Ricci scalar

$$R = \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \ . \tag{3.11}$$

Combining these expressions with the energy-momentum tensor (3.7) we can particularize the Einstein's equations (3.4) to the homogenous and isotropic case. We obtain the so-called Friedmann equations

$$H^2 = \frac{\rho}{3M_P^2} + \frac{\Lambda}{3} - \frac{k}{a^2}\,, \tag{3.12}$$

$$\dot{H} + H^2 = -\frac{1}{6M_P^2}(\rho + 3p) + \frac{\Lambda}{3}\,, \tag{3.13}$$

with the dots denoting derivatives with respect to the coordinate time t and

$$M_P = (8\pi G)^{-1/2} = 2.436 \times 10^{18}\ \text{GeV} \tag{3.14}$$

the *reduced Planck mass*. The quantity

$$H \equiv \frac{\dot{a}}{a} \tag{3.15}$$

is the so-called *Hubble rate*. This parameter is positive for an expanding Universe and negative for a contracting one. The value of the Hubble parameter at the present epoch is the Hubble constant

$$H_0 = 100\,h\ \text{km}\,\text{s}^{-1}\text{Mpc}\,, \tag{3.16}$$

with $h = 67.8 \pm 0.9$ and Mpc$= 3 \times 10^{24}$ cm standing for "megaparsec". This value allows us to estimate the present age and size of the Univere

$$H_0^{-1} = 9.77\,h^{-1}\,\text{Gyr} \qquad\qquad cH_0^{-1} = 3000\,h^{-1}\,\text{Mpc}\,. \tag{3.17}$$

The Friedmann equations (3.12) and (3.13) can be combined to obtain the *continuity equation*

$$\dot{\rho} + 3H\,(\rho + p) = 0. \tag{3.18}$$

Exercise

Derive Eq. (3.18) i) by combining Eqs. (3.12) and (3.13) and ii) from the covariant energy-momentum conservation (3.5). Interpret the result by considering the adiabatic dilution of energy due to the expansion and the work done by pressure.

Hint: Consider the second law of thermodynamics $TdS = dU + pdV$.

The cosmological evolution following from Eqs. (3.12), (3.13) and (3.18) can be determined once a pressure to energy density relation $p(\rho)$ is specified. We will restrict ourselves to *barotropic fluids* for which the pressure is linearly proportional to the energy density,

$$p = w\rho\,, \tag{3.19}$$

with w the so-called *equation-of-state parameter*. This case covers the two main matter components in the hot Big Bang scenario, namely (non-relativistic) matter ($w = 0$) and radiation ($w = 1/3$).

Exercise

Consider a macroscopic collection of structureless point particles interacting through spatially localized collisions. On distances d much larger than the typical mean free path, the number of particles is large and the statistical fluctuations about the mean properties of the fluid are expected to be small. If the fluid is isotropic,[a] the mean density and pressure observed by a comoving observer over the volume $\Delta V = d^3$ can be written as

$$\rho = \left\langle \sum_n E_n \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V}, \qquad p = \frac{1}{3} \sum_i \left\langle \sum_n p_n^i v_n^i \delta^{(3)}\left(\mathbf{x} - \mathbf{x}_n\right) \right\rangle_{\Delta V}, \tag{3.20}$$

with $E_n = \sqrt{\mathbf{p}_n^2 + m_n^2}$ the energy of the individual particles. The index i is a space index ranging from 1 to 3 and n selects the particle of mass m_n and momentum $\mathbf{p}_n$. Use these microscopic expressions to derive the equation of state for non-relativistic matter and radiation.

[a] i.e if the fluid is *perfect*.

In our our Universe, several species with different equations of state coexist. Their relative contribution is traditionally parametrized by the dimensionless parameters

$$\Omega_{\rm M} \equiv \frac{\rho_{\rm M}}{3M_P^2 H^2}, \qquad \Omega_{\rm R} \equiv \frac{\rho_{\rm R}}{3M_P^2 H^2}, \qquad \Omega_\Lambda \equiv \frac{\Lambda}{3M_P^2 H^2}, \qquad \Omega_{\rm K} \equiv -\frac{k}{(aH)^2}, \tag{3.21}$$

with the subindices M, R, Λ and K standing for matter, radiation, cosmological constant and curvature contributions. At present time, the radiation and curvature contributions are very small ($\Omega_{\rm R} \simeq 5 \times 10^{-5}$, $\Omega_{\rm K} < 0.005$) and

$$\Omega_{\rm M} \simeq 0.3\,, \qquad \Omega_\Lambda \simeq 0.7\,, \tag{3.22}$$

Our present Universe is therefore dominated by a cosmological constant or dark energy component. Note however that is was dominated by matter and radiation in the past. This can be easily seen by considering the scaling of non-relativistic matter and radiation. Integrating the conservation equation (3.18) and using Eq. (3.12) for the zero curvature case ($k = 0$), we get

$$\rho \propto a^{-3(1+w)}, \qquad a(t) \propto \begin{cases} t^{2/3(1+w)} & w \neq -1\,, \\ e^{Ht} & w = -1\,. \end{cases} \tag{3.23}$$

For non-relatistic matter ($w = 0$), the energy density dilutes with the volume $\rho_M \sim a^{-3}$, reflecting mass conservation. For relativistic matter ($w = 1/3$), the energy density dilutes as $\rho_R \sim a^{-4}$, due to the additional redshift of energy ($\propto a^{-1}$). Note that the radiation domination period cannot be eternal to the past. When $t \to 0$, the scale factor goes to zero and the physical energy density ρ diverges.

3.2 Troubles in paradise

In spite of the success of the hot Big Bang for describing the observed Universe, it is not free of problems.

3.2.1 Flatness problem

Consider the dimensionless *energy density parameter*

$$\Omega \equiv \frac{\rho}{\rho_{\rm crit}}\,, \tag{3.24}$$

with

$$\rho_{\rm crit} \equiv 3M_P^2 H^2 \tag{3.25}$$

the so-called *critical energy density*. In terms of this quantity, the Friedmann equation (3.12) becomes

$$\Omega - 1 = \frac{k}{(aH)^2}\,. \tag{3.26}$$

The quantity $\Omega - 1$ measures the curvature of the Universe. A Universe with flat spacial sections ($k = 0$) corresponds to $\Omega = 1$. For $k \neq 0$, the evolution of the curvature depends on the evolution of the *comoving Hubble radius* $(aH)^{-1}$. If the Hubble radius $(aH)^{-1}$ increases/decreases with time, the curvature increases/decreases accordingly. In a Universe dominated by a fluid with equation of state w, the comoving Hubble radius evolves as

$$(aH)^{-1} \propto a^{\frac{1}{2}(1+3w)}\,. \tag{3.27}$$

For standard matter sources satisfying the *strong energy condition* $1+3w > 0$, $(aH)^{-1}$ grows as the Universe expands. For instance, during matter (MD) and radiation domination (RD) we have

$$(aH)^{-1} \propto a^{1/2}\,, \quad \text{(MD)} \qquad\qquad (aH)^{-1} \propto a\,. \quad \text{(RD)} \tag{3.28}$$

Exercise
Derive Eq .(3.27).

The density parameter Ω at present time is very close to one. Specifically, the latest Planck satellite data combined with baryon acoustic oscillations (BAO) give

$$\Omega_0 - 1 = 0.000 \pm 0.005\,, \tag{3.29}$$

at the 95% C.L. Taking into account this value ($\Omega_0 \sim 10^{-3}$) together with the evolution equations (3.28) for the comoving Hubble radius during matter and radiation domination, we can compute the value of $\Omega - 1$ at the time of matter-radiation equality ($z_{\rm eq} = 3600$)

$$\Omega(z_{\rm eq}) - 1 = \Big(\Omega_0 - 1\Big)(1 + z_{\rm eq})^{-1} \approx 2.8 \times 10^{-5}\,, \tag{3.30}$$

and at Big Bang Nucleosynthesis ($z_{\rm BBN} = 10^{10}$)

$$\Omega(z_{\rm BBN}) - 1 = (\Omega(z_{\rm eq}) - 1)\left(\frac{1+z_{\rm eq}}{1+z_{\rm BBN}}\right)^2 \approx 3.6 \times 10^{-18}\,. \tag{3.31}$$

A percent deviation from flatness in the present Universe translates into *unnaturally* small deviations at early epochs. In others words, in order to recover the Universe we observe

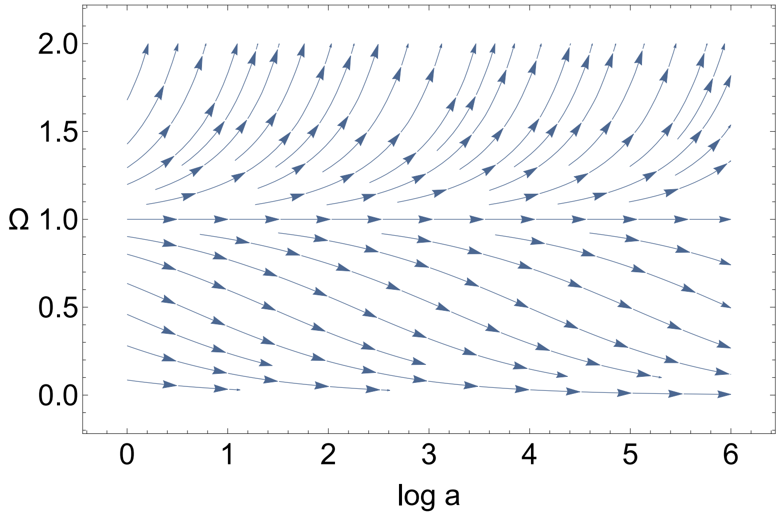

Figure 3.4: Evolution of the energy density parameter Ω in standard cosmology. The point $\Omega = 1$, corresponding to flat curvature, is a repeller.

today the initial conditions must be terribly fine-tuned. Any deviation from these initial conditions translates either into a closed and recollapsing Universe or into an open Universe completely dominated by curvature. This extreme dependence on the initials conditions is highly unsatisfactory.

Exercise

One could argue that naturalness is just a question of taste and that the most symmetric initial conditions are somehow more natural. However, this is not very convincing from the point of view of the self-consistency of the theory, specially if those initial conditions are unstable. Show that the dimensionless energy density parameter satisfies the differential equation

$$\frac{d\Omega}{d\log a} = (1+3w)\Omega(\Omega - 1)\,. \tag{3.32}$$

Note that for both matter ($w = 0$) and radiation ($w = 1/3$) we have $1 + 3w > 0$, meaning that a flat Universe with $\Omega = 1$ is an *unstable* fixed point. If $\Omega > 1$ at some point of the evolution, it will keep on growing; and viceversa, if $\Omega < 1$ at some point, it will keep on decreasing. This behaviour is illustrated in Fig. 3.4.

3.2.2 Horizon problem

Our Universe seems to be extremely homogeneous on large scales. The CMB temperature anisotropies arise only at the level of one part in 10^5. However, if the Universe were only radiation- or matter-dominated in the past, any two regions in the sky with angular separation of a few arc degrees would not have been able to communicate between the singularity and recombination to decide which the common temperature is supposed to be. This is the so-called horizon problem. To discuss this problem let us start describing the causal structure

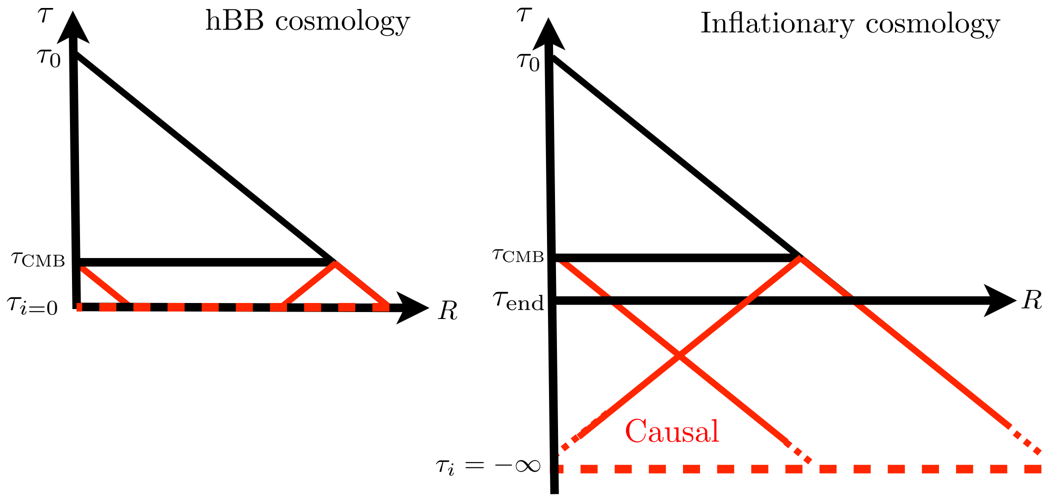

Figure 3.5: Conformal diagram for standard hBB cosmology (left) and inflationary cosmology (right). Inflation solves the horizon problem by extending the conformal diagram to negative conformal times.

of the FLRW metric. Performing a change of coordinates

$$dR = \frac{dr}{\sqrt{1-kr^2}}\,, \qquad r = S_k(R) = \begin{cases} \sin R & \text{for } k=1\,, \\ R & \text{for } k=0\,, \\ \sinh R & \text{for } k=-1\,, \end{cases} \,, \tag{3.33}$$

the FLRW metric (3.2) can be written as

$$ds^2 = -dt^2 + a(t)^2\left[dR^2 + S_k^2(R)(d\theta^2 + \sin^2\theta d\phi^2)\right]\,. \tag{3.34}$$

Defining now the conformal time

$$\tau = \int \frac{dt}{a(t)}, \tag{3.35}$$

we can recast Eq. (3.34) in the conformal form

$$ds^2 = a(\tau)^2\left[-d\tau^2 + dR^2 + S_k^2(R)(d\theta^2 + \sin^2\theta d\phi^2)\right]\,. \tag{3.36}$$

The causal structure of the FLRW metric is determined by the way in which light propagates on null geodesics with $ds^2 = 0$. Since the space is isotropic, we can freely set the coordinates θ and ϕ to a constant value. In this case, the condition $ds^2 = 0$ implies

$$R = \pm\tau + \text{constant}\,. \tag{3.37}$$

The fact that geodesics are 45° lines in the $\{\tau, R\}$ plane is related to the fact that the the FLRW metric (3.34) is conformally flat. If the Universe started at some initial time t_i, then there is a maximum amount of time for light to have travelled. The *(comoving) particle* *horizon* is the largest distance that a photon can travel between t_i and a later time $t > t_i$ (recall that $c \equiv 1$)

$$d_H = \tau - \tau_i = \int_{t_i}^{t} \frac{dt}{a} = \int_{a_i}^{a} \frac{da}{a\dot{a}} = \int_{\ln a_i}^{\ln a} (aH)^{-1} d\ln a \,. \tag{3.38}$$

According to this expression, the evolution of d_H depends also on the evolution of the *comoving Hubble radius* $(aH)^{-1}$ (cf. Eq. (3.27)).

Exercise

Show this.

For standard matter sources $1 + 3w > 0$ and $(aH)^{-1}$ grows as the Universe expands. When that happens. the integral in Eq. (3.38) becomes *dominated by its upper limit*

$$d_H \propto \frac{2}{(1+3w)} a^{\frac{1}{2}(1+3w)} = \frac{2}{(1+3w)} (aH)^{-1} \,. \tag{3.39}$$

Note that due to the presence of the singularity at $a_i = 0$ (or equivalently at $\tau_i = 0$) this quantity is finite. At each instant of time, regions that were never in causal contact before get into contact for the first time. The fact that two of these regions share approximately the same temperature cannot by a consequence of thermal equilibrium. On general grounds, these regions should be expected to look very different from each other. This applies also to the CMB (see the left pannel of Fig. 3.5). The observed homogeneity of the CMB map is not only remarkable but also strange and unexpected! Most points in that map share roughly the same temperature even if the naive horizon scale at decoupling is just a few arc degrees. How is this possible?

3.3 Inflationary paradigm

The horizon and flatness problems are intimately related to the fact that the comoving Hubble radius $(aH)^{-1}$ grows within the standard hot Big Bang scenario. A simple solution to these problems is to postulate a decrease of $(aH)^{-1}$ at early times

$$\frac{d}{dt}(aH)^{-1} < 0 \,. \tag{3.40}$$

or equivalently a violation the strong energy condition $1 + 3w > 0$ (cf. Eq (3.27)). This additional phase in the history of the Universe is called *inflation*. The name can be easily understood by noticing that Eq. (3.40) implies *accelerated expansion*

$$\frac{d}{dt}(aH)^{-1} = \frac{d}{dt}(\dot{a})^{-1} = -\frac{\ddot{a}}{(\dot{a})^2} < 0 \quad \Rightarrow \quad \ddot{a} > 0. \tag{3.41}$$

No weak energy condition violation

Note that the condition $\ddot{a} > 0$ is very different from $\dot{H} > 0$ with

$$\dot{H} = \frac{\dot{a}}{a} - H^2 = -\frac{1}{2M_P^2}(\rho + p)\,. \tag{3.42}$$

In an expanding Universe the energy density is always decreasing or at most constant. In order to have $\dot{H} > 0$, the null energy condition $\rho + p \geq 0$ should be violated.

If the inflationary stage lasts long enough, the hot Big Bang problems are automatically solved. In particular, if the comoving Hubble radius $(aH)^{-1}$ in Eq. (3.26) decreases, the curvature is driven towards zero (the unstable point $\Omega = 1$ in Eq. (3.32) becomes now an attractor). This solves the flatness problem. On the other hand, if $1 + 3w < 0$ the integral in Eq. (3.38) becomes *dominated by its lower limit* and the singularity is pushed towards negative conformal times,

$$\tau_i \propto \frac{2}{(1+3w)} a_i^{\frac{1}{2}(1+3w)} = -\infty\,. \tag{3.43}$$

The extension of the conformal diagram to negative conformal times allows the light cones of widely separated CMB points to intersect (see the right pannel of Fig. 3.5). This solves the horizon problem.

3.3.1 Minimal duration of inflation

Due to the Hubble shrinking, typical scales that were inside the horizon at the onset of inflation, leave the radius of causal contact as inflation proceeds. When inflation ends, the comoving Hubble radius $(aH)^{-1}$ starts increasing and the scales reenter the horizon. This is illustrated in Fig. 3.6. A simple inspection of this figure reveals that in order to solve the hot Big Bang problems we must require

$$(a_i H_i)^{-1} \geq (a_0 H_0)^{-1}\,, \tag{3.44}$$

with $(a_i H_i)^{-1}$ the comoving Hubble radius at the onset of the inflationary regime and $(a_0 H_0)^{-1}$ its value today. Let us assume for simplicity that the radiation-dominated epoch starts immediately after the end of inflation and neglect the comparatively shorter matter and dark-energy dominated epochs. Under these assumptions, we have

$$\frac{(a_i H_i)^{-1}}{(a_{\rm end} H_{\rm end})^{-1}} = \frac{(a_i H_i)^{-1}}{(a_0 H_0)^{-1}} \frac{(a_0 H_0)^{-1}}{(a_{\rm end} H_{\rm end})^{-1}} \geq \frac{a_0}{a_{\rm end}}\,, \tag{3.45}$$

where in the last step we have made use of the condition (3.44) together with the radiation domination scaling $H \propto a^{-2}$. Taking into account that $a_0/a_{\rm end} \sim T_{\rm end}/T_0$ and assuming inflation to finish at an energy scale[1] $T_{\rm end} \sim 10^{14}\,{\rm GeV} \sim 10^{26} T_0$, with $T_0 \sim 10^{-3}$ eV the

[1] Although accurate enough for a large set of inflationary models, this value is taken for illustration purposes only. On general grounds, the precise number of e-folds must be computed model by model taking into account not only the energy scale at the end of inflation, but also the details of the reheating process and the effects of any intermediate era between the end of inflation and the onset of radiation domination.

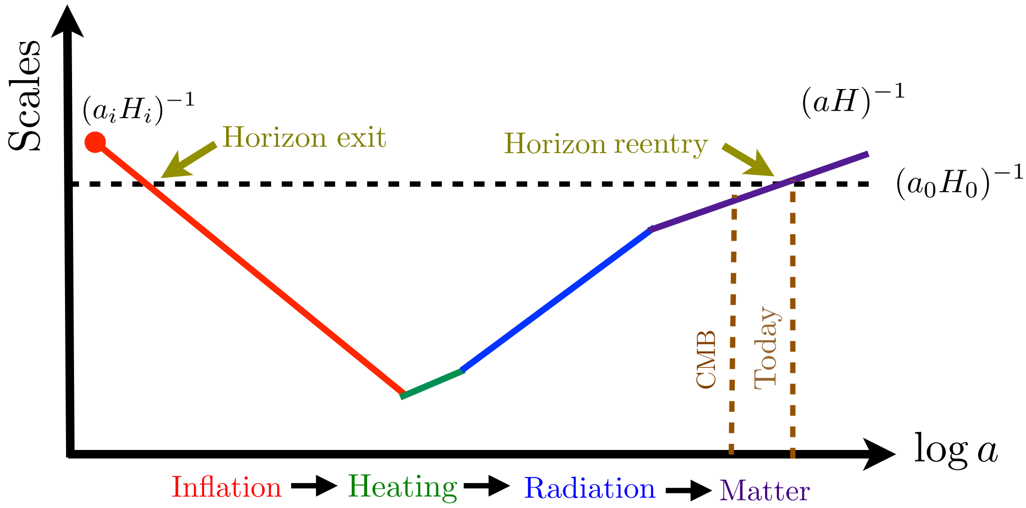

Figure 3.6: Scales of cosmological interest as a function of the number of e-folds. Due to the Hubble shrinking, typical scales $\lambda \equiv (a_0 H_0)^{-1}$ that were inside the horizon at the onset of inflation, leave the radius of causal contact as inflation proceeds. When inflation ends, the comoving Hubble radius $(aH)^{-1}$ starts increasing and the scales reenter the horizon.

temperature of the Universe today, we get

$$\frac{(a_i H_i)^{-1}}{(a_{\text{end}} H_{\text{end}})^{-1}} \geq 10^{26} \simeq e^{60} \,. \tag{3.46}$$

For $H_i \simeq H_{\text{end}}$ (see below), this conditon becomes

$$N \equiv \log\left(\frac{a_{\text{end}}}{a_i}\right) \gtrsim 60 \,. \tag{3.47}$$

Therefore, in order to solve the inflationary problems we need, *at least*, $N = 60$ *e-folds* of inflation.

3.3.2 Hubble flow parameters

The conditions for inflation are traditionally formulated as conditions on the variation of the Hubble rate. Defining the fractional change of the Hubble rate per e-fold[2] N,

$$\epsilon \equiv -\frac{\dot{H}}{H^2} = -\frac{d\ln H}{dN} \,, \qquad \mathrm{d}N \equiv H\mathrm{d}t = \mathrm{d}\ln a \,, \tag{3.48}$$

we can rewrite Eq. (3.40) as

$$\frac{d}{dt}(aH)^{-1} = -\frac{1}{a}(1-\epsilon) \,. \tag{3.49}$$

[2]This quantity will play a central role in the effective field theory of inflation to be presented in Chapter 7.

For inflation to take place, ϵ must be smaller than one. As argued in Section 3.3.1, the solution of the flatness and horizon problems requires a rather long inflationary stage. In order to achieve this, the fractional change of ϵ,

$$\eta \equiv \frac{d\ln\epsilon}{dN} = \frac{\dot{\epsilon}}{H\epsilon}\,, \tag{3.50}$$

must also be small, $|\eta| < 1$. Note that the η parameter are just the first two elements of a full series of *Hubble flow parameters*

$$\epsilon_{i+1} \equiv \frac{d\ln\epsilon_i}{dN} = \frac{\dot{\epsilon}_i}{H\epsilon_i}\,. \tag{3.51}$$

3.3.3 de Sitter spacetime

The minimal value of the ϵ parameter is zero. In this case the Hubble rate H is constant and the Universe expands exponentially fast, $a(t) = e^{Ht}$. This limit motivates the study of the so-called de Sitter spacetime.

The de Sitter spacetime dS_4 can be represented as a 4-dimensional hyperboloid *extrinsically* embedded in a d=5 Minkowski spacetime $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1)$ with coordinates z^A

$$\eta_{AB}z^A z^B = -z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = l^2\,. \tag{3.52}$$

The quantity $l \equiv 1/H$ is the so-called *de Sitter radius.*[3] This representation makes explicit the symmetries of the de Sitter space: rotations and Lorentz transformations in the 10 planes formed by pairs of the five coordinates z^A. This ten parameter $SO(4,1)$ group plays the same instrumental role than the Poincare group in Minkowski spacetime. In particular, it greatly facilitates computations as far as quantum field theory is concerned.

The dS_4 spacetime can be also described in an *intrinsic* way. Consider the transformation

$$z_0 = \frac{1}{H}\sinh(Ht) + \frac{H}{2}e^{Ht}\delta_{ij}x^i x^j\,, \tag{3.53}$$

$$z_i = x_i e^{Ht}\,, \tag{3.54}$$

$$z_4 = \frac{1}{H}\cosh(Ht) - \frac{H}{2}e^{Ht}\delta_{ij}x^i x^j\,, \tag{3.55}$$

with $i = 1,2,3$ and $-\infty < t < \infty$ and $-\infty < x_i < \infty$. In this coordinate system the line element (3.52) becomes a special case of the *flat* FLRW spacetime

$$ds^2 = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \tag{3.56}$$

with $a(t) = e^{Ht}$. Note however that Eq. (3.56) is not completely equivalent to (3.52), since the coordinates $\{t, x^i\}$ cover only half of de Sitter manifold. This can be easily seen by adding the z_0 and z_4 coordinates (see also Fig. (3.7)),

$$z_0 + z_4 = \frac{1}{H}e^{Ht} \geq 0\,. \tag{3.57}$$

[3] The choice of notation will become clear soon.

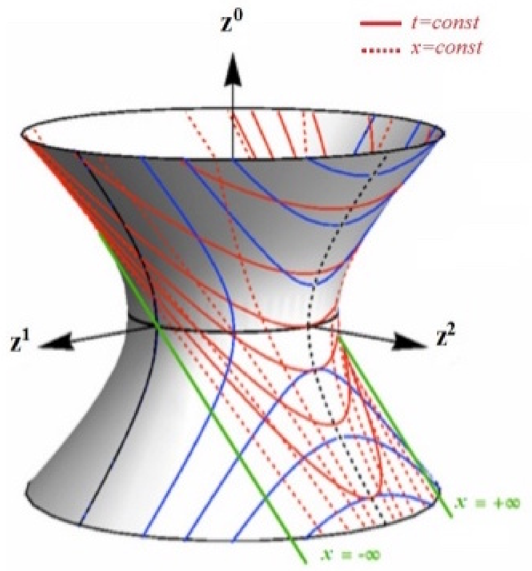

Figure 3.7: The embedding of de Sitter space into a five dimensional flat space-time with two spatial coordinates suppressed. The flat coordinates in (3.53)-(3.55) cover only half of de Sitter manifold. The surfaces (lines) of constant t and constant **x** are also indicated.

Exercise

1. Derive Eq. (3.56) from the 5-dimensional embedding (3.52).
2. Other choices of coordinates leading to FLRW metrics with open and closed spatial sections can be also considered. Find these sets of coordinates.

It is interesting to recast (3.56) in terms of the conformal time (3.35). Taking into account that

$$\tau = -\frac{1}{H}e^{-Ht} \qquad \Longrightarrow \qquad a(\tau) = -\frac{1}{H\tau}\,, \tag{3.58}$$

the line element takes the manifestly conformally flat form

$$ds^2 = \frac{1}{H^2\tau^2}\left(-d\tau^2 + \delta_{ij}dx^i dx^j\right)\,, \tag{3.59}$$

with η ranging between $-\infty$ and 0. Note that Eq. (3.59) is manifestly invariant under the rescaling

$$\tau \to \lambda\tau\,, \qquad x^i \to \lambda x^i\,. \tag{3.60}$$

As we will see in Section 6.1.3, this symmetry plays a central role in the properties of the primordial perturbations generated during inflation. But let not anticipate things and focus on the background evolution for the time being. What seems clear is that in order to recover the hot Big Bang scenario the de Sitter phase cannot be eternal. In other words, we need to equip the de Sitter space *with a clock*.