Unruh effect

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1 Introduction

In normal quantum field theories (QFT's) one deals with observers in inertial frames, which means that Lorentz invariant quantities are conserved even if one changes the frame to another inertial frame. This report is going to consider the case of accelerated observers. This will lead to the Unruh effect (Unruh, 1976) which predicts particle production of a field due to the accelerated observer.
To begin the Rindler space-time will be defined as the frame belonging to a uniformly accelerated observer after which the Unruh effect for a toy model of a 2-dimensional massless scalar field will be derived. This part will follow the work of Mukhanov's and Winitzki's book "Introduction to Quantum Effects in Gravity". After this the case of more general motions will be briefly discussed.

2 Rindler space-time

2.1 Uniformly accelerated motion

In the classical case uniformly accelerated motion is defined as $\frac{d\mathbf{v}}{dt} = \text{const.}$ where $\mathbf{v} = \frac{d\mathbf{x}}{dt}$ is the velocity in the lab frame. When one however considers a relativistic setting this definition would lead to $v = |\mathbf{v}|$ becoming larger than c. Therefore in a relativistic case one should find a different definition for uniformly accelerated motion. By defining $\mathbf{a} = \text{const.}$ in the proper frame that is the frame belonging to the accelerated observer one has achieved an alternative definition.
An accelerated observer does not have a single comoving frame but a whole set of them each for a particular proper time point τ. These comoving frame

as well as the lab frame are all inertial frames which means that squares of 4-vectors are conserved. Thus

$$u^\mu u_\mu = 1 \quad \text{and} \quad a^\mu a_\mu = -a^2 \tag{1}$$

where $u^\mu(\tau), a^\mu(\tau)$ are the respective 4-vectors of the velocity and acceleration and $a = |\mathbf{a}|$ in a particular comoving frame. This implies that $a^0 = 0$ which can be found by taking the derivative with respect to τ of the equation involving the velocity in the same comoving frame.
Identifying $a^\mu = \frac{du^\mu}{d\tau}$ and setting $\mathbf{a} = (a, 0, 0)$ with $a > 0$ leads to the following coupled differential equations

$$(u^0)^2 - (u^1)^2 = 0 \tag{2}$$
$$(\partial_\tau u^0)^2 - (\partial_\tau u^1)^2 = -\, a^2 \tag{3}$$

where all non-relevant directions have been dropped. Solving the equations be decoupling leads to

$$u^0(\tau) = \cosh(a\tau) \tag{4}$$
$$u^1(\tau) = \sinh(a\tau) \tag{5}$$

which describes the velocity of the observer in the lab frame. Integrating over τ leads us to the trajectory described by

$$t(\tau) = t_0 + \frac{1}{a}\sinh(a\tau) \tag{6}$$
$$x(\tau) = x_0 - \frac{1}{a} + \frac{1}{a}\cosh(a\tau) \tag{7}$$

One can now set the initial conditions $t_0 = 0,\ x_0 = \frac{1}{a}$ to specify on which hyperbolae the observer travels. As seen in Figure 1 the observer is not causally connected to every point in space-time.

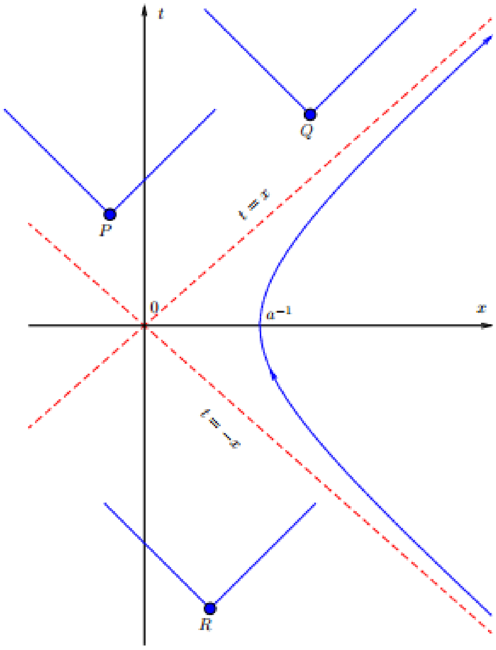

Figure 1: Trajectory of an uniformly accelerated observer. Taken from Mukhanov & Winitzki "Introduction to Quantum Effects in Gravity"

2.2 Line element

To write down the line element of the proper frame which will henceforth be called Rindler frame one needs to define the coordinates $(\tau,\ \xi)$ in it as functions of the lab frame coordinates $(t,\ x)$. This can be done by describing the endpoint of a rigid stick moving along with the observer, who will be considered to move along the line with $\xi = 0$, in the lab frame and use that point to define the coordinates in Rindler spacetime.
In a particular comoving frame the stick is given by $s_c^\mu = \binom{0}{\xi}$. To describe this in the lab frame one just needs to apply an inverse Lorentz boost with the velocity $v = \frac{u^1}{u^0}$ which yields

$$s_l^\mu = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} s_c^\mu = \begin{pmatrix} u^0 & u^1 \\ u^1 & u^0 \end{pmatrix} s_c^\mu = \begin{pmatrix} u^1\xi \\ u^0\xi \end{pmatrix}. \tag{8}$$

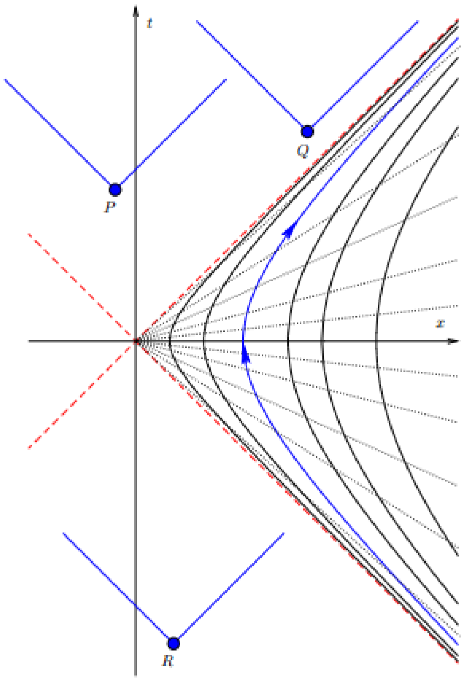

Figure 2: Rindler space embedded in Minkowski space. Taken from Mukhanov & Winitzki "Introduction to Quantum Effects in Gravity"

To describe the endpoint of the stick traveling with an accelerated observer one therefore just needs to add s_l^μ to the trajectory of the observer giving

$$t(\tau,\ \xi) = \frac{1 + a\xi}{a} \sinh(a\tau) \tag{9}$$

$$x(\tau,\ \xi) = \frac{1 + a\xi}{a} \cosh(a\tau). \tag{10}$$

The inverse transform then reads

$$\tau = \frac{1}{2a} \log(\frac{x + t}{x - t}) \quad \text{and} \quad \xi = -\frac{1}{a} + \sqrt{x^2 - t^2} \tag{11}$$

where one can see that ξ is only defined in the interval $[\frac{1}{a},\ \infty)$ and therefore does not describe the whole space but only a part of it. This is the reason why one often also finds the Rindler space referred to as Rindler wedge. Indeed, one can see that the Rindler space only describes the part of the lab frame bounded by $x = |t|$ as one sees in Figure 2 where lines of constant ξ are given by hyperbolae and lines of constant τ are straight lines through the origin.

To obtain the line elemnet one now takes the total differential of (9) and (10) given by

$$dt = (1 + a\xi)\cosh(a\tau)d\tau + \sinh(a\tau)d\xi \tag{12}$$

$$dx = (1 + a\xi)\sinh(a\tau)d\tau + \cosh(a\tau)d\xi \tag{13}$$

and inserts this into the line element leading to

$$ds^2 = dt^2 - dx^2 = (1 + a\xi)^2 d\tau^2 - d\xi^2 \tag{14}$$

which is the Rindler line element. This can be written in a conformally flat way by applying the transform $\tilde{\xi} = \frac{1}{a}\log(1 + a\xi)$ to obtain

$$ds^2 = e^{2a\tilde{\xi}}(d\tau^2 - d\tilde{\xi}^2). \tag{15}$$

3 Two dimensional scalar field

To derive the Unruh effect a two dimensional massless scalar field will be used as a toy model and described in the lab frame and the Rindler space. The action for this field is given by

$$S = \frac{1}{2}\int dx^2 \sqrt{-g} g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi \tag{16}$$

in the general case of an arbitrary metric. In the lab frame the metric is simply Minkowskian whereas Rindler space-time is obtained by a conformal transformation which leave the action invariant in two dimension (see Appendix A). In this case the action reads

$$S_M[\Phi] = \frac{1}{2}\int dt\ dx\ (\partial_t\Phi)^2 - \partial_x\Phi)^2 \tag{17}$$

$$S_R[\Phi] = \frac{1}{2}\int d\tau\ d\xi\ (\partial_\tau\Phi)^2 - \partial_\xi\Phi)^2 \tag{18}$$

where the coordinate $\tilde{\xi}$ has been replaced with ξ. This has the usual equations of motion

$$\partial_t^2\Phi - \partial_x^2\Phi = 0 \tag{19}$$

$$\partial_\tau^2\Phi - \partial_\xi^2\Phi = 0 \tag{20}$$

which has the general solution of a right- and left-moving function

$$\Phi(t, x) = A(t - x) + B(t + x) \tag{21}$$

$$\Phi(\tau, \xi) = P(\tau - \xi) + Q(\tau + \xi) \tag{22}$$

with A, B, P, Q some arbitrary smooth functions.
One can quantise the field using the usual mode expansion leading to two different pairs of creation and annihilation operators

$$\hat{\Phi}(t,x) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} (e^{-i|k|t+ikx} \hat{a}_k^- + e^{i|k|t-ikx} \hat{a}_k^+) \tag{23}$$

$$\hat{\Phi}(\tau,\xi) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \frac{1}{\sqrt{2|k|}} (e^{-i|k|\tau+ik\xi} \hat{b}_k^- + e^{i|k|\tau-ik\xi} \hat{b}_k^+) \tag{24}$$

which in generally are not identical. The annihilation operators define the respective vacuum states in the usual way

$$\hat{a}_k^- |0_M\rangle = 0 \quad \text{and} \quad \hat{b}_k^- |0_R\rangle = 0 \quad \forall k \tag{25}$$

which are completely different physical states. The respective operators should be related to another via some general Bogulyubov transform.

3.1 Lightcone coordinates

To find this transform it is beneficial to change to lightcone coordinates defined by

$$u = \tau - \xi,\ v = \tau + \xi,\ \bar{u} = t - x,\ \bar{v} = t + x \tag{26}$$

which is sensible as the general solution to Φ will now decouple into parts depending only on one variable

$$\Phi(\bar{u},\bar{v}) = A(\bar{u}) + B(\bar{v}) \tag{27}$$

$$\Phi(u,v) = P(u) + Q(v). \tag{28}$$

As the two frames were related to another via coordinate transforms one can still express the lab frame with Rindler coordinates in the following way

$$\bar{u} = -\frac{1}{a} e{-}au, \quad \bar{v} = \frac{1}{a} e^{av}. \tag{29}$$

This will be very useful later when one wants to relate the two mode expansions with each other. By switching to lightcone coordinates the mode expansion will read

$$\hat{\Phi}(\bar{u},\bar{v}) = \int_0^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+ + e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+) \tag{30}$$

$$\hat{\Phi}(u,v) = \int_0^{\infty} \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} (e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+ + e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+) \tag{31}$$

where the variable k has been renamed to ω or Ω which is going to describe the momenta. Here one can see that the mode expansion also decouples into parts depending on u and v allowing us to identify the parts defining A, B or P, Q

$$A(\bar{u}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega\bar{u}} \hat{a}_\omega^- + e^{i\omega\bar{u}} \hat{a}_\omega^+) \tag{32}$$

$$B(\bar{v}) = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} (e^{-i\omega\bar{v}} \hat{a}_{-\omega}^- + e^{i\omega\bar{v}} \hat{a}_{-\omega}^+) \tag{33}$$

$$P(u) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} (e^{-i\Omega u} \hat{b}_\Omega^- + e^{i\Omega u} \hat{b}_\Omega^+) \tag{34}$$

$$Q(v) = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} (e^{-i\Omega v} \hat{b}_{-\Omega}^- + e^{i\Omega v} \hat{b}_{-\Omega}^+). \tag{35}$$

As the field should still be the same independent of the frame one is choosing to describe it and $A(\bar{u}) = A(\bar{u}(u))$, the left- and right-moving terms must be equal to each other therefore

$$A(\bar{u}(u)) = P(u) \tag{36}$$

must hold.

3.2 Bogulyubov transformation

To find the relation linking the operators to each other one must evaluate (36). This will be done by applying a Fourier transform to both sides of the equation to find

$$\int_{-\infty}^\infty \frac{du}{\sqrt{2\pi}} e^{*\Omega u} P(u) = \frac{1}{\sqrt{2|\Omega|}} \begin{cases} \hat{b}_\Omega^- & , \Omega > 0 \\ \hat{b}_{|\Omega|}^+ & , \Omega < 0 \end{cases} \tag{37}$$

for the right-hand side of the equation. The left-hand side reads

$$\int_{-\infty}^\infty \frac{du}{\sqrt{2\pi}} e^{*\Omega u} A(\bar{u}) = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} \int_{-\infty}^\infty \frac{du}{2\pi} (e^{i\Omega u - i\omega\bar{u}} \hat{a}_\omega^- + e^{i\Omega u + i\omega\bar{u}} \hat{a}_\omega^+) \tag{38}$$

$$= \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} (F(\omega, \Omega) \hat{a}_\omega^- + F(-\omega, \Omega) \hat{a}_\omega^+) \tag{39}$$

with

$$F(\omega, \Omega) = \int_{-\infty}^\infty \frac{du}{2\pi} \exp(i\Omega u + i\frac{\omega}{a} e^{-au}) \tag{40}$$

an integral related to the Gamma function. By substituting $x \equiv e^{-au}$ one obtains

$$F(\omega,\Omega) = \frac{1}{2\pi a}\int_0^\infty dx\; x^{-\frac{i\Omega}{a}-1} e^{\frac{i\omega}{a}x} \tag{41}$$

which is an integral of the form

$$\int_0^\infty dx\; x^{s-1} e^{-bx} = b^{-s}\Gamma(s) \tag{42}$$

The only problem here is that $s = -\frac{i\Omega}{a}$ and $b = -\frac{i\omega}{a}$ are complex numbers and the integral convergence depends on the real value of s and b.
As $\Re(s) = 0$ the limit of $\Re(b) \to 0^+$ can only be understood in a distributional sense. To obtain the limit one needs to take the limit of b and s having a vanishing real part larger than 0. Therefore the transform

$$s = -\frac{i\Omega}{a} + \epsilon \tag{43}$$

$$b = -\frac{i\omega}{a} + \epsilon, \quad \epsilon > 0 \tag{44}$$

in the limit of $\epsilon \to 0$ is used. One can then rewrite the integral (42)

$$\int_0^\infty dx\; x^{s-1} e^{-bx} = b^{-s}\Gamma(s) = \exp(-s\log b)\Gamma(s) \tag{45}$$

where one must only evaluate the logarithm of b. This is done by choosing the branch

$$\log(a+ib) \equiv \log|a+ib| + i\ \mathrm{sign}(b)\arctan\left(\frac{|b|}{a}\right), \quad a > 0 \tag{46}$$

to obtain

$$\log b = \lim_{\epsilon\to+0}\log\left(-\frac{i\omega}{a}+\epsilon\right) = \log\left|\frac{\omega}{a}\right| - i\frac{\pi}{2}\mathrm{sign}\left(\frac{\omega}{a}\right) \tag{47}$$

which after inserting into (45) gives us

$$F(\omega,\Omega) = \frac{1}{2\pi a}\exp\left[\frac{i\Omega}{a}\log\left|\frac{\omega}{a}\right| + \frac{\pi\Omega}{2a}\mathrm{sign}\left(\frac{\omega}{a}\right)\right]\Gamma\left(-\frac{i\Omega}{a}\right). \tag{48}$$

By comparing the two Fourier transforms one finds the relation between $\hat{b}^-_\Omega$ and all $\hat{a}^\pm_\omega$ as

$$\hat{b}^-_\Omega = \int_0^\infty d\omega[\alpha_{\omega\Omega}\hat{a}^-_\omega + \beta_{\omega\Omega}\hat{a}^+_\omega] \tag{49}$$

with

$$\alpha_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(\omega, \Omega) \tag{50}$$

$$\beta_{\omega\Omega} = \sqrt{\frac{\Omega}{\omega}} F(-\omega, \Omega); \quad \omega, \Omega > 0. \tag{51}$$

The operator $\hat{b}^{+}_{\Omega}$ can then be found as the Hermitian conjugate. This relation is as expected a Bogulyubov transform that mixes the creation and annihilation operators with different momenta. The transform for negative momenta Ω is obtained by the same procedure for $B(\bar{v}(v)) = Q(v)$.

3.3 Number density

As the two vacuum states $|0_M\rangle, \ |0_R\rangle$ defined by the operators $\hat{a}^{\pm}_{\omega}$ and $\hat{b}^{\pm}_{\Omega}$ are physically different, one expects a-particles in the Rindler vacuum and vice versa. To check this, the expectation value of a b particle number operator given by $\hat{N}_{\Omega} \equiv \hat{b}^{+}_{\Omega}\hat{b}^{-}_{\Omega}$ is computed in the Minkowski vacuum

$$\langle \hat{N}_{\Omega} \rangle \equiv \langle 0_M | \hat{b}^{+}_{\Omega}\hat{b}^{-}_{\Omega} | 0_M \rangle \tag{52}$$

$$= \langle 0_M | \int_0^{\infty} d\omega [\alpha^{*}_{\omega\Omega}\hat{a}^{+}_{\omega} + \beta^{*}_{\omega\Omega}\hat{a}^{-}_{\omega}] \int_0^{\infty} d\omega' [\alpha_{\omega'\Omega}\hat{a}^{-}_{\omega'} + \beta_{\omega'\Omega}\hat{a}^{+}_{\omega'}] | 0_M \rangle \tag{53}$$

$$= \int_0^{\infty} d\omega |\beta_{\omega\Omega}|^2. \tag{54}$$

During this computation one of the integrals cancels due to the commutator, of $\hat{a}^{-}_{\omega}$ giving a δ function, which arises from commutating the annihilation operator to act on the vacuum.
To evaluate the last integral one can first rewrite the commutator relation for $\hat{b}^{\pm}_{\Omega}$:

$$\delta(\Omega - \Omega') = [\hat{b}^{-}_{\Omega}, \hat{b}^{+}_{\Omega}] \tag{55}$$

$$= \int_0^{\infty} d\omega (\alpha_{\omega\Omega}\alpha^{*}_{\omega\Omega'} - \beta_{\omega\Omega}\beta^{*}_{\omega\Omega'}) \tag{56}$$

If one then takes (48) one finds that

$$F(\omega, \Omega) = \exp\left(\frac{\pi\Omega}{A}\right) F(-\omega, \Omega) \tag{57}$$

which can be used to rewrite the coefficients $\alpha_{\omega\Omega}$ to obtain

$$\delta(\Omega - \Omega') = \int_0^{\infty} d\omega \frac{\Omega}{\omega} F^{*}(-\omega, \Omega) F(-\omega, \Omega) \left[\exp\left(\frac{\pi\Omega + \pi\Omega'}{a}\right) - 1\right]^{-1} \tag{58}$$

this can then be inserted for $|\beta_{\omega\Omega}|^2$ in (54) to finally find

$$\langle \hat{N}_\Omega \rangle = \int_0^\infty d\omega |\beta_{\omega\Omega}|^2 = \int_0^\infty d\omega \frac{\Omega}{\omega} |F(-\omega, \Omega)|^2 \tag{59}$$

$$= \left[\exp\left(\frac{2\pi\Omega}{a} \right) - 1 \right]^{-1} \delta(0) \equiv n_\Omega \delta(0) \tag{60}$$

where $\delta(0)$ describes the volume factor of the whole space whereas n_Ω describes the corresponding density.
This results reminds of a Bose distribution

$$n(E) = \left[\exp\left(\frac{E}{T} \right) - 1 \right]^{-1} \tag{61}$$

which allows us to define the Unruh temperature

$$T_U \equiv \frac{a}{2\pi}. \tag{62}$$

This is the Unruh effect which states that accelerated observers produce particles or in different interpretation behave as if they were placed in a thermal bath of temperature T_U. In principle it is possible to detect particles of any energy although it is increasingly improbable the higher the energy.

3.4 Detection

It is notable that the Unruh effect is a very inefficient process to produce particles. This can easily be seen by calculating the factor $\frac{c}{\hbar k_B}$ which needs to be multiplied to T_U to restore SI-units

$$\frac{c}{\hbar k_B} \approx 2.55 * 10^{-20} \frac{s^2 K}{m}. \tag{63}$$

It follows that the acceleration necessary to achieve the equivalent of $1K$ needs to be of the order $10^{20} \frac{m}{s^2}$ which shows that one would need huge amounts of energy to source the acceleration.
At this point it is important to note that the Unruh effect does not need the validation of explicit experimental proof similar to the case of the free scalar field in quantum field theory.

4 General motions

If one considers more general motions instead of uniformly accelerated motion one often loses the ability to state the metric of the proper frame.Thus

it becomes difficult to derive the effect as done above.
To solve this one now considers the field Φ coupled to a detector moving along a world line $x_\mu(\tau)$ and then calculates the response function for an excitation of the detector which is equated to particle production when viewed from the Minkowski space (Obadia & Milgrom, 2007).
In general one often loses the thermal characteristics when leaving uniformly accelerated motion but still observes particle detection coming from the various vacua. This is why, in some parts of the literature, the Unruh effect is only understood to regard uniformly accelerated motion.

5 Discussion

To summarise the Unruh effect states that an accelerated observer will observe particles in a thermal distribution. Viewed from the other side though, inertial observers detect that the accelerated observer produces particles as it follows it'S world line.
Some physicists prefer to drop the concept of particle detection and creation from the effect altogether due to the associated conceptual problems and prefer to think of it from a purely statistical mechanics point of view which couples the detector to an infinite reservoir at thermal equilibrium and thus evolves to equilibrium itself, giving rise to the Unruh effect.
Disregarding whether one approves of the particle concept or not the energy necessary to drive the Unruh effect is understood to come from the same agent as that which sources the acceleration and is usually regarded as an infinite pool.

References

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