Euclidean Field Theory: revisiting Unruh, Hawking and de Sitter

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Abstract

This report wants to discuss the relation between periodicity in imaginary time and thermodynamics. We analyze the case of a complex scalar field and derive the Green function for it both at zero and finite temperature. Then we prove that they can be analytically continued to the complex plane and that they show a periodicity, that is related to the temperature of the state on which they are computed. We apply then these results to recover Unruh effect, Hawking radiation and De Sitter temperature.

1 Introduction

Studying Green's functions it turns out that they can be analytically extended to the complex plane and that finite temperature Green functions are periodic in imaginary time. In a general relativistic framework one can provide a interesting interpretation of this periodicity since it is strictly related to the temperature of the state on which they are computed. Actually one can also give a new definition of equilibrium state by requiring KMS condition to be fulfilled. The usefulness of this formalism is that one can avoid the steps of analytic continuation+imaginary axis restriction, by starting directly from the "Euclideanized" problem, that means starting from a Wick rotation of the time coordinate. By doing that one can directly see the period of the time coordinate and derive the temperature from that. This is very quick and so this is why this method is the most commonly used.

Note: In the following we will consider $c = \hbar = G = k_B = 1$

2 Thermodynamical properties and periodicity

2.1 Brief derivation of Eulidean two point function

A scalar field ϕ on a generic static [1] manifold $\Re\times$M, where $\Re$ is the real axis and M is a Riemanian n-dimensional manifold with metric $ds^2 = \gamma_{ij}dx^i dx^j$, is defined as the solution of

$$\left(\frac{\partial^2}{\partial t^2} + K\right)\phi = 0 \tag{1}$$

Where K is define in this space to be

$$K = -\gamma^{jk}(x)[\nabla_j - iA_j(x)][\nabla_k - iA_k(x)] + V(x)$$

With V(x) an external potential and A(x) a gauge boson field.

If ψ_ν are the normalized eigenfunctions of K (that means $K\psi_\nu = \omega_\nu^2\psi_\nu$) one can write:

$$\phi(x,t) = \sum_{\nu=1}^{\infty}\psi_\nu(2\omega_\nu)^{-1/2}[exp(-i\omega_\nu t)a_\nu + exp(i\omega_\nu t)b_\nu^\dagger]$$

where a_ν annihilates a quantum and $b_\nu^\dagger$ creates an antiquantum[2] $[a_\nu, a_\mu^\dagger] = \delta_{\mu\nu}$

The two point functions for a field ϕ are defined as:

$$G_+^\infty \equiv \langle 0|\phi(t_2,x)\phi^\dagger(t_1,y)|0\rangle = \langle 0|\phi(t,x)\phi^\dagger(0,y)|0\rangle = \sum_{\nu=1}^{\infty}\psi_\nu\psi_\nu^*(2\omega_\nu)^{-1}exp(-i\omega_\nu t)$$

$$G_-^\infty \equiv \langle 0|\phi^\dagger(t_1,x)\phi(t_2,y)|0\rangle = \sum_{\nu=1}^{\infty}\psi_\nu\psi_\nu^*(2\omega_\nu)^{-1}exp(-i\omega_\nu t)$$

Where the "∞" denotes that the vacuum where the function are calculated is at zero temperature ($\beta = \infty$).

Now let's investigate the analytic continuation of these functions on the complex plane and introduce at this purpose the complex variable $z = t + is$. Both $G_\pm^\infty$ are holomorphic functions in the upper and in the lower half plane respectively. It can be shown that with the use of the complex variable z $G_+^\infty(z,x,y) = G_-^\infty(z,x,y)$ for z in a certain interval $-d < t < d$ on the real

[1]That means in generale that its metric: $\partial g_{ij}/\partial t = 0, g_{0j} = 0$

[2]That means that ϕ as an operator field acts on a certain Fock space that is built starting from a specific vacuum vector $|0\rangle$ and this is completely specified by the Green two point function.

axis.[3] Thus (because of the one dimensional edge-on-the-wedge theorem)[4] each of this function results to be the analytic continuation of the other, that means that there is a unique holomorphic function that can be defined:

$$\mathcal{G}^{\infty}(z,x,y) = \begin{cases} G^{\infty}_{+}(z,x,y) & \text{Im}z < 0, \\ G^{\infty}_{-}(z,x,y) & \text{Im}z > 0. \end{cases}$$

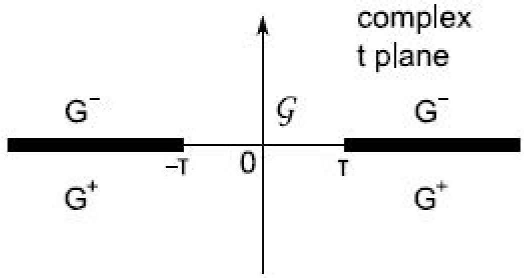

Figure 1: Cut plane where is defined continuation $\mathcal{G}$ of $G^{\pm}$ [from [2]]

Now consider the restriction to the imaginary axis of $\mathcal{G}$, that is to say to set $z = is$. It turns out:

$$G^{\infty}(s,x,y) \equiv \mathcal{G}^{\infty}(is,x,y) = \sum_{\nu=1}^{\infty} \psi_\nu(x)\psi_\nu(y)^*(2\omega_\nu)^{-1} exp(-\omega_\nu|s|)$$

that can also be written (if $s_2 - s_1 = s$):

$$G^{\infty}(s,x,y) = (2\pi^{-1}) \sum_{\nu=1}^{\infty} \int_{-\infty}^{\infty} dk_0 \psi_\nu(x) exp(ik_0 s_2) \psi_\nu(y)^* exp(-ik_0 s_1)(k_0^2 + \omega_\nu^2)^{-1}$$

That's the *integral kernel* of the inverse of the operator $-\partial^2/\partial s^2 + K$, or equivalently one can say that G^{∞} is the Green function of it acting on $\Re \times$M, that is to say:

$$\left(-\frac{\partial^2}{\partial s_2^2} + K_{(x)} \right) G^{\infty} = \delta(s_2 - s_1)\delta(x-y)\gamma(y)^{-1/2}$$

So $G^{\infty}(s,x,y) \equiv G^{\infty}_{E}$ is the Green function of the "Euclideanized" problem. Note that now the operator is elliptic and hence has a unique Green function

[3] In fact the difference between these two function is $G^{\infty}_{+}(t,x,y) - G^{\infty}_{-}(t,x,y) = \langle 0|[\phi(t,x),\phi^{\dagger}(0,y)]|0\rangle = [\phi(t,x),\phi^{\dagger}(0,y)]$ is the commutator that vanishes for sufficiently small time, so there must be a region around t=0 where G^{+} and G^{-} match

[4] Holomorphic functions on two "wedges" with an "edge" in common are analytic continuations of each other provided they both give the same continuous function on the edge. Suppose that f is a continuous complex-valued function on the complex plane that is holomorphic on the upper half-plane, and on the lower half-plane. Then it is holomorphic everywhere.

and that by analytic continuation one can reach one of the two Green functions $G^{\infty}_{\pm}(t,x,y)$ of the initial problem that are solutions instead of

$$\left(\frac{\partial^2}{\partial t_2^2}+K_{(x)}\right)G^{\infty}_{\pm}=0$$

One further property: The Euclidean Green function is also related to the the time ordered two point function, that is the one that defines the Feynman propagators:

$$G^{\infty}_F(t,x,y)\equiv\langle|0|\mathbf{T}[\phi(t_2,x)\phi^{\dagger}(t_1,y)]|0\rangle=\begin{cases}G^{\infty}_{+}(t,x,y) & \text{if } t_2>t_1\\ G^{\infty}_{-}(t,x,y) & \text{if } t_2<t_1\end{cases}$$

one can see that this can be obtained by rigid rotation of G^{∞} from s to t axis:

$$G^{\infty}_F=(t,x,y)=G^{\infty}(-it,x,y)=\sum_{\nu=1}^{\infty}\psi_{\nu}(x)\psi_{\nu}(y)*(2\omega_{\nu})^{-1}exp(_{i}\omega_{\nu}|t|)$$

2.2 Thermal properties

We learned how to compute Green functions in the ground state, and found out:

$$G(x,x')\equiv\langle\Omega|\phi(x)\phi(x')|\Omega\rangle=\sum_{\nu}\frac{1}{2\omega_{\nu}}\psi_{\nu}(x)\psi^{*}_{\nu}(x')e^{-i\omega_{\nu}(t-t')}$$

One can not only be interested in the ground state but also in the state that has a certain temperature $T\neq 0$. In general one can define a thermal expectation value of an operator A as:

$$\langle A\rangle_{\beta}=\frac{Tr(e^{-\beta H}A)}{Tr(e^{-\beta H})}=\mathcal{Z}^{-1}Tr(e^{-\beta H}A) \qquad (2)$$

where $\mathcal{Z}$ is the *partition function* and $\beta=1/T$ (if we set $k_B=1$). One can show that:

$$\frac{\mathrm{Tr}e^{-\beta\omega a^{*}a}aa^{*}}{\mathrm{Tr}e^{-\beta\omega a^{*}a}}=\frac{1}{1-e^{-\beta\omega}}$$

$$\frac{\mathrm{Tr}e^{-\beta\omega a^{*}a}a^{*}a}{\mathrm{Tr}e^{-\beta\omega a^{*}a}}=\frac{1}{e^{\beta\omega}-1}$$

So one finds that the thermal Green function:

$$G_{\beta}(t,x,t',x')=\sum_{j\nu}\frac{1}{2\omega_{\nu}}\frac{\psi_{\nu}(x)\psi^{*}_{\nu}(x')e^{-i\omega_{\nu}(t-t')}}{1-e^{-\beta\omega_{\nu}}}+\frac{\psi^{*}_{\nu}(x)\psi_{\nu}(x')e^{i\omega_{\nu}(t-t')}}{e^{\beta\omega_{\nu}}-1}$$

Or:

$$G^{\beta}=\sum_{\nu}\frac{1}{2\omega_{\nu}}\frac{\psi_{\nu}\psi^{*}_{\nu}e^{-i\omega_{\nu}(t-t')}}{1-e^{-\beta\omega_{\nu}}}\left(e^{-i\omega_{\nu}(t-t')}+e^{-\beta\omega_{\nu}}e^{i\omega_{\nu}(t-t')}\right)$$

That is the integral kernel of the operator

$$(2\sqrt{K})^{-1}(1-e^{-\beta\sqrt{K}})^{-1}(e^{-i\sqrt{K}(t-t')}+e^{-\beta\sqrt{K}}e^{i\sqrt{K}(t-t')})$$

K is allowed here to have a non discrete spectrum. Then we claim a new criterion of existence of thermal equilibrium states, that is $C_0^\infty(\Sigma)\subset\mathcal{D}(K^{1/2})$. The states constructed then are no more Gibbs state, but they rather fulfill the more general **KMS condition**.

KMS condition :

The Kubo, Martin, Schwinger condition is commonly accepted as a general definition of thermal equilibrium state at temperature $T=\beta^{-1}$ in quantum field theory. This reads as:

$$G_-^\beta(z,A,B)=G_+^\beta(z-i\beta,A,B)$$

or more generally:

$$\langle BA_t\rangle_\beta=\langle A_{t-i\beta}B\rangle$$

Let's see where these come from. Assume to have an arbitrary quantum-mechanical system with a time-independent hamiltonian H. Let's have A an observable operator, that then evolves in the Heisemberg picture as:

$$A_t\equiv e^{itH}Ae^{-itH}$$

Now remeber the definition of thermal state in (2) to compute:

$$\begin{aligned}G_+^\beta(t,A,B)\quad&\equiv\langle A_tB\rangle_\beta\\&=\mathcal{Z}^{-1}Tr[e^{-\beta H}e^{itH}Ae^{-itH}B]\\&=\mathcal{Z}^{-1}Tr[e^{-\beta H}Ae^{-itH}Be^{itH}]\\&=\langle AB_{-t}\rangle_\beta\end{aligned}$$

where we made use of the ciclicity of the trace ($Tr(ABC)=Tr(CAB)=Tr(BCA)$) and of the fact that two operator that are function of H commute.

Similarly one defines:

$$G_-^\beta(t,A,B)=\langle BA_t\rangle_\beta=...=\langle B_{-t}A\rangle_\beta$$

Therefore:

$$G_+^\beta(t,A,B)=G_-^\beta(-t,B,A)$$

In the complex plane, defining $z=t+is$, we have:

$$G_+^\beta(z,A,B)=\mathcal{Z}^{-1}Tr[e^{i(z+i\beta)H}Ae^{-izH}B]$$

$$G_-^\beta(z,A,B)=\mathcal{Z}^{-1}Tr[Be^{izH}Ae^{-i(z-i\beta)H}]$$

Then replacing z with $z-i\beta$ one finds

$$G_-^\beta(z,A,B)=G_+^\beta(z-i\beta,A,B)$$

And so recovering the previous definition we conclude: $\langle BA_t\rangle_\beta=\langle A_{t-i\beta B}\rangle_\beta$

2.3 Periodicity

So one found for the thermal equilibrium states that:

$$G_\beta^\pm(t,x,x') = \sum_\nu \frac{\psi_\nu \psi_\nu^*(x')}{2\omega_\nu(1-e^{-\beta\omega_\nu})} q_\pm(t)$$

Where we collected the factor by defining:

$$q_\pm(z) = e^{\mp i\omega_\nu z} + e^{\pm i\omega_\nu(z \pm i\beta)}$$

And so $G_\beta^\pm$ are analytic functions in $-\beta < \mathrm{Im}z < 0$ or $0 < \mathrm{Im}z < \beta$ respectively. Like their ground state pairs, they can be continued in a single holomorphic function $\mathcal{G}_\beta(z,x,x')$ on the cut plane $\{z \in \mathbf{C} : |\mathrm{Im}z| < \beta\} \backslash ((-\infty,-\tau) \cup (\tau,\infty))$

From the definition one sees an important statement of periodicity:

$$q_+(z - i\beta) = q_-(z)$$

for any z, and that allows us to say that

$$\mathcal{G}_\beta(z,x,x') = \mathcal{G}_\beta(z + iN\beta, x, x')$$

for all integer N

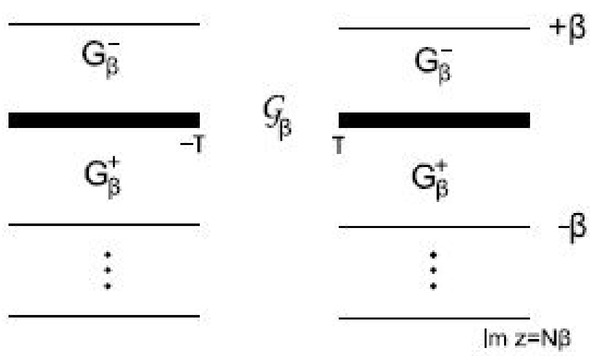

Figure 2: Periodicity of the thermal two-point functions in the cut plane [from [2]]

Similarly to the case above we claim that the restriction to the imaginary axis $\mathcal{G}_\beta(i(s-s'),x,x')$ is the Euclidean Green function on $L^2(\mathbf{T}_\beta \times \Sigma)$ where $\mathbf{T}_\beta$ is a circle of periodicity β. One can prove that by verifying that

$$\frac{1}{2\omega_j(1-e^{-\beta\omega_j})}\left(e^{-\omega_j|s-s'|}+e^{\omega_j(|s-s'|-\beta)}\right)$$

Is the Green function of the operator $\frac{-\partial^2}{\partial s^2}+\omega_j^2$ on $L^2(\mathbf{T}_\beta)$

Here β is not just a random period but has a deep meaning since it defines the thermal properties of the state on which the Green function is computed. So that's it: if one can find that the system has some periodicity on its coordinate, then can state that also the Green function has a periodicity and the period directly drives to the temperature. So what people do is to study the Euclideanized problem and derive temperature from the period in the imaginary time.

3 Application of the periodicity properties in horizon related effects

3.1 Unruh effect

An accelerated detector in Minkowski spacetime ($ds^2 = dt^2 - dx^2 - dx_{i\perp}^2$) sees thermal spectrum of particle in Minkowski vacuum state. This phenomenon is named **Unruh effect**. Let's start from defining new coordinates that describes the Rindler's wedge:

$$t = r\sinh\tau, \quad x = r\cosh\tau \tag{3}$$

where x is chosen to be the direction of the acceleration of the observer and $(\tau, r) \in \Re \times (0,\infty)$ (These boundaries cover a special region of spacetime known as *Rindler wedge* (**R**) defined by the condition $|t| < x^n$.) With those coordinates the usual Minkowski metric (1,-1,-1,-1) turns into:

$$ds^2 = r^2 d\tau^2 - dr^2 - \delta_{ij}dx_\perp^i dx_\perp^j \tag{4}$$

Here we want to consider the effect of this background on the theromodynamical properties of quantum field, so let's take into account a scalar field ϕ and its equation of motion. The Klein-Gordon equation in this set of coordinates reads:

$$\frac{\partial^2\phi}{\partial\tau^2} + K\phi = 0 \tag{5}$$

with [5]

$$K = -(r^2\partial_r^2 + r\partial_r + r^2\triangle_\perp) + m^2r^2$$

Now to derive the Unruh effect is sufficient to show that the Green function that belongs to a thermal state of $\beta = 2\pi$ in this region (that means that one

[5] K is derived as in section 2.1 with Rindler's metric

compute the two point function starting from (5)) is the same as the one of a Minkoskian space but computed in a ground state if one makes use of polar coordinates.

$$\begin{array}{lll} \mathcal{G}_R^{2\pi} & & \mathcal{G}_M^{\infty} \\ \downarrow & & \downarrow \\ G_R^{2\pi} & \longleftrightarrow & G_M^{\infty} \\ (r,\sigma) & & (s,x) \end{array}$$

Admitting that the Green functions can be continued to the complex plane one can find:

$$\mathcal{G}_R^{2\pi}(\zeta,(\mathbf{r_x},\mathbf{x}_\perp),(\mathbf{r_y},\mathbf{y}_\perp)) \qquad \zeta=\tau+i\sigma$$

and then compare it with

$$\mathcal{G}_M^{\infty}(z,\mathbf{x},\mathbf{y}) \qquad z=t+is$$

One can find out that the Minkowski Green function has this behavior[6]:

$$\mathcal{G}_M^{\infty}(z,\mathbf{x},\mathbf{y})=F(-z^2+|\mathbf{x}-\mathbf{y}|^2)$$

With $F(w)$ a holomorphic function except for $w=0$ (that is not a problem if one restrict to Rindler's wedge). Changing coordinates as in (3) one gets:

$$\mathcal{G}_M^{\infty}(t_2-t_1,\mathbf{x},\mathbf{y})=F(r_x^2+r_y^2-2r_xr_y\cosh(\tau_2-\tau_1)+|\mathbf{x}_\perp-\mathbf{y}_\perp|^{\mathbf{2}})$$

Once restricted to the complex plane this function can be shown to be the integral Kernel of the operator

$$\left(\frac{-\partial^2}{\partial\sigma^2}+K^2\right)^{-1}$$

on $L^2(S^1\times M;r^{-1}d\sigma dr d\Omega)$. This restriction is nothing but the Euclidean Green function $G_M^{\infty}(s_2-s_1,\mathbf{x},\mathbf{y})$ in polar coordinates: $s=r\sin\sigma$, $x=r\cos\sigma$

Thus one can prove that:

$$\left[\frac{-\partial^2}{\partial\sigma_2^2}+K\right]G_R^{2\pi}(\sigma_2-\sigma_1,(r_x,\mathbf{x}_\perp),(r_y,\mathbf{y}_\perp))=\delta(\sigma_2-\sigma_1)\delta(r_x-r_y)\delta(\mathbf{x}_\perp-\mathbf{y}_\perp)r_y$$

Is the transcription to polar coordinates of

$$\left[\frac{-\partial^2}{\partial s_2^2}-\nabla^2+m^2\right]G_M^{\infty}(s_2-s_1,\mathbf{x},\mathbf{y}))=\delta(s_2-s_1)\delta(\mathbf{x}-\mathbf{y})$$

Then one can conclude that the two states are equivalent: a Minkoswki vacuum state is equivalent to a thermal state of $\beta=2\pi$ if seen with Rindler's coordinates.

[6]For the demonstration see [1]

The same result can be obtained automatically if one starts from the Euclidean problem. Let's perform the **Wick rotation** on the metric in (4) and set: $\tau = i\tau_E$ and $t = i\sigma$. Then automatically the Euclideanized Minkowski coordinates are periodic with period 2π in τ_E (in fact $t = r\sinh\tau = \sinh i\tau_E = i\sin\tau_E$). The new metric has now the form:

$$ds^2 = -dr^2 + -r^2 d\tau_E^2 - \delta_{ij} dx_\perp^i dx_\perp^j = -ds_E^2 \tag{6}$$

Thermal equilibria states correspond exactely to the Euclidean Green function defined with this metric. Remember that $\tau_E \in (0, \beta)$ is periodically identified with $\beta = 2\pi$ in this case. Using then Rindler coordinates we found a periodicity in the imaginary time and we linked it to the temperaure of the equilibrium state: $T = (2\pi)^{-1}$. Notice that this temperature is related to the observer since is only the coordinates relative to the accelerated observer that present this periodicity.

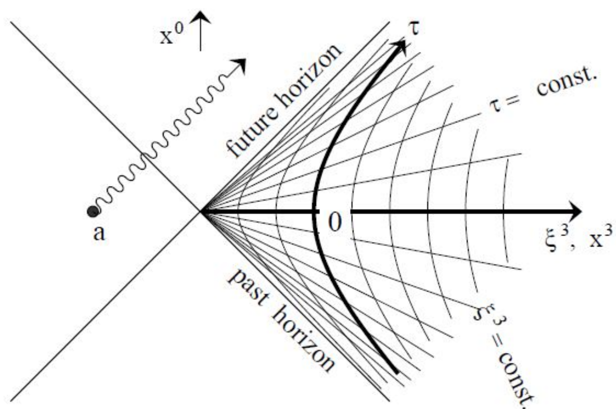

Figure 3: Rindler spacetime. Hyperbola are lines of constant acceleration. The line $t = x^0 = \pm x$ are respectively the future and the past horizon

With the proper normalization it turns out that the temperature is actually:

$$T = \frac{1}{2\pi r}$$

and hence diverges as $r \to 0^+$ that means as the observer stays on curves on constant acceleration closer to the horizon.

3.2 Horizon temperature of Schwarzschild black hole

Reconsidering the *Hawking radiation* one can figure out that the thermal character of the radiation from black holes can be explained by the fact that the black hole metrics can be analytically continued to "Euclidean", i.e. positive definite, metrics which are periodic in the imaginary time coordinate $\tau = it$ with period β. This means that any fields which are analytic in the real Lorentzian black hole metric are periodic in the imaginary time coordinate and so behave as if they were at finite temperature $T = \beta^{-1}$. Let's see it for a **Schwarzschild black hole** (that is a static and uncharged black hole of mass M), described by the metric:

$$-ds^2 = -\left(1 - \frac{2M}{r}\right)dt^{*2} + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

with: $d\Omega^2 = d\theta^2 + sin^2\theta d\phi^2$

At r=2M the radial coefficient of the metric diverges: here we have the **horizon**. Remember that this metric is valid outside the BH, hence in the interval $2M < r < \infty, -\infty < t^* < \infty$. However this singularity in the metric can be pulled away if one consider the Regge-Wheeler, or *tortoise*, coordinates:

$$r^* = r + 2M \ln(r/2M - 1)$$

now the radial coordinate has no more boundaries: $-\infty < r^* < \infty$, and the resulting metric is:

$$-ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^{*2} + dr^{*2}) + r^2 d\Omega^2$$

Now let's set $\tau \equiv t^*/4M$ and $r' \equiv 2Me^{\rho} \equiv 2Me^{r^*/4M} = 2M[(r-2M)]^{1/2}e^{r/4M}$, and perform the coordinate change: s

$$t = r' \sinh(\tau),\ x = r' \cosh(\tau)$$

to get:

$$-ds^2 = \frac{8M}{r}e^{-r/2M}(-dt^2 + dx^2) + r^2 d\Omega^2$$

that is called **Kruskal metric**.

Note that here r depends on t, thus the metric is not static and because of that it is not possible to define a vacuum in the usual sense. However Kruskal space has an analytical extension to a complex manifold. This can be seen by the transformation:

$$\frac{8M}{r}e^{-r/2M}(ds^2+dx^2+r^2d\Omega^2) = \left(1-\frac{2M}{r}\right)ds^{*2}+\left(1-\frac{2M}{r}\right)^{-1} dr^2+r^2d\Omega^2 \quad (7)$$

where $s^* = 4M\sigma, \quad \zeta = \tau + i\sigma$

The only way to have (4) to be analytic is to admit that s^* is periodic with period $\beta = 8\pi M$. It is sufficient to consider the Euclideanized metric:

$$ds_E^2 = \left(\frac{2M}{r}\right)d\tau^2 + \left(\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

and see what happens in the vicinity of the horizion, let's say at $r = 2M + \epsilon$, $\epsilon \to 0$. We have:

$$ds_E^2 = \left(1 - \frac{2M}{R}\right)^{-1} \left(\frac{\epsilon^2}{(4M)^2} d\tau^2 + d\epsilon^2\right) + r^2 d\Omega^2$$

The only way to make the divergence disappear is to set the term into brackets at zero, that leads to a differential equation for $\tau/4M$ that then turns out to be a complex exponential that has periodicity of 2π. So it is natural to consider that there must exists a state on which one can define an analytic Green function and that would be a state of $T_M = (8\pi M)^{-1}$.

3.3 De Sitter space-time

The close connection between event horizon and thermodynamics can be extended to cosmological model with repulsive cosmological constant. Let's consider here the case of a **De Sitter ($d\mathcal{S}$) model** i.e. a model with non-zero curvature and non-zero cosmological constant. $d\mathcal{S}$-spacetime can be viewed as a 4-d hyperboiloid embedded in a 5-d Minkowski spacetime ($\eta_{AB} = diag(1, -1, -1, -1, -1)$):

$$\mathcal{S}_\Lambda : \eta_{AB}\xi^A\xi^B = -R^2, \quad R^2 \equiv 3\Lambda^{-1}$$

where Λ is the *cosmological constant*.
If we simply consider the Euclidean coordinate by setting $\xi^0_* = i\xi^0$, we recover to (changing the overall signs):

$$\mathcal{S}^4 : (\xi^0_*)^2 + (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 + (\xi^4)^2 = R^2$$

That is a 4-dimensional sphere, hence its coordinates are automatically periodic of period $\beta = 2\pi R$. The temperature thus is

$$T = \frac{1}{2\pi R}$$

then is strictly related to the cosmological properties of the space (radius of curvature (R) and cosmological constant Λ). The interesting thing is that the temperature here is a proprerty of the spacetime itself and does not depend on the observer as it was instead in Rindler space.

4 Suggestion for further readings:

1 Stephen A. Fulling and Simon N. M. Ruijsenaars. *Temperature, periodicity and horizons.* Physics reports, 152(3):135176, 1987

2 Christopher J. Fewster, *Lectures on quantum field theory in curved spacetime*, 2008

3 Chethan Krishnan. *Quantum field theory, black holes and holography.* arXiv preprint arXiv:1011.5875, 2010.