Master Seminar: The Heat Kernel Computation

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1 Motivation: Effective action

The aim of this paper is to calculate the effective action of a scalar field in a slightly curved spacetime. For this purpose we will develop in section **??** the procedure of Zeta functions and heat kernels. We will go through the calculation of the heat kernel to leading order in curvature in section **??**. In the present section we will introduce the definition and basics of the effective action, and we will briefly summarize some of the main points of interest of this functional.

1.1 Definition of effective action

The effective action is defined as a functional $\Gamma_L[J(t)]$ through the relation

$$e^{i\Gamma_L[J(t)]} = \int_{q(-\infty)=0}^{q(+\infty)=0} \mathcal{D}q \, e^{iS[q(t),J]} \tag{1.1}$$

Here, the subindex L stands for Lorentz signature (i.e. $ds^2 = dt^2 - d\mathbf{x}^2$), we will shortly see the relevance of this remark. The r.h.s. of equation (1.1) is the propagator for a quantized system, $K(q_f, q_0, t_f, t_0)$ with boundary conditions $q_f(t_f \to \infty) = q_0(t_0 \to \infty) = 0$ and which interacts with an external field $J(t)$. $S[q(t), J]$ is the classical Lagrangian action (classical in the sense that no gravitational effects are yet taken into consideration). For the sake of example we write the action of a driven harmonic oscillator as:

$$S[q(t), J(t)] = \int_{t_0}^{t_f} \mathrm{d}t \left[\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + J(t)q \right] \tag{1.2}$$

Nevertheless, a mathematically rigorous definition of functional integration is only available for Euclidean path integrals, that is, the decaying analog of the oscillating Lorentzian exponential equation 1.1, which is expected to converge better and is easier to perform. Thus, we perform a Wick rotation $t = -i\tau$ so that the action 1.2 is expressed as:

$$iS[q(t), J(t)]_{t=-i\tau} = -\int_{\tau_0}^{\tau_f} \mathrm{d}\tau \left[\frac{1}{2}\dot{q}^2 + \frac{1}{2}\omega^2 q^2 - J(t)q \right] \equiv -S_E[q(\tau), J(\tau)] \tag{1.3}$$

where we now wrote $\dot{q} \equiv \mathrm{d}q/\mathrm{d}\tau$, and the path integral becomes

$$\int_{q(-\infty)=0}^{q(+\infty)=0} \mathcal{D}q \, e^{-S_E[q(\tau), J(\tau)]} \equiv e^{-\Gamma_E[J(\tau)]} \tag{1.4}$$

which defines the Euclidean effective action, $\Gamma_E[J(\tau)]$. The calculations made in the following sections will then involve this form of the effective action.

To conclude this section, we motivate the definition use of the Lorentzian effective action, which we would obtain from analytic continuation of the Euclidean analog back to the Lorentzian time. It is clear from equation 1.2, that we can split the classical (Lorentzian) action into the homogeneous and inhomogeneous contributions:

$$S[q, J] = S_0[q] + \int J(t)q(t) \, \mathrm{d}t$$

where the integral boundary conditions have been omitted for clarity. In general we can always split the $J(t)$-dependent term from the homogeneous contribution of the action up to a total derivative. This means that the classical action is linearly dependent on the external field, so we can write

$$\frac{\delta S[q, J]}{\delta J(t_1)} = q(t_1)$$

using functional derivative notation. This implies

$$\int \mathcal{D}q\, q(t_1) e^{iS[q,J])} = \frac{1}{i}\frac{\delta}{\delta J(t_1)}\int \mathcal{D}q\, e^{iS[q,J]} = \frac{1}{i}\frac{\delta}{\delta J(t_1)} e^{i\Gamma_L[J]}$$

so we can express the one-point correlation function as

$$\frac{\langle 0|\hat{q}(t_1)|0\rangle}{\langle 0|0\rangle} = \frac{\int \mathcal{D}q\, q(t_1) e^{iS[q,J]}}{\int \mathcal{D}q\, e^{iS[q,J]}} = \frac{1}{e^{i\Gamma_L[J]}}\frac{\delta}{i\delta J(t_1)} e^{i\Gamma_L[J]} = \frac{\delta \Gamma_L[J]}{\delta J(t_1)} \tag{1.5}$$

Hence, the effective action acts as a generating functional for n-points correlation functions:

$$\int \mathcal{D}q\, q(t_1) q(t_2) \ldots q(t_n) e^{iS[q,J]} = \frac{\delta}{i\delta J(t_1)}\frac{\delta}{i\delta J(t_2)} \cdots \frac{\delta}{i\delta J(t_n)}\int \mathcal{D}q\, e^{iS[q,J]}$$

As a last note, we state without proof the following property of the effective action in a curved spacetime: its variation with respect to the metric tensor gives the vacuum expectation value of the energy-momentum tensor $T_{\mu\nu}$ in the following manner:

$$\frac{1}{2}\sqrt{-g(x)}\langle 0|\hat{T}_{\mu\nu}(x)|0\rangle = \frac{\delta \Gamma_L}{\delta g_{\mu\nu}(x)}$$

2 Zeta functions and heat kernels

The aim of the following sections is to construct the effective action of a quantum scalar field, $\phi(x)$ in a gravitational metric, $g_{\mu\nu}$. We will consider a general number of dimensions 2ω. Under the natural assumption that the field vanishes at $x \to \infty$, the classical lagrangian action can be rewritten as

$$S[\phi, g_{\mu\nu}] = \frac{1}{2}\int \mathrm{d}^{2\omega}x\, \sqrt{-g}\, [g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(x)\phi^2] = \frac{1}{2}\int \mathrm{d}^{2\omega}x\, \sqrt{-g}\, \phi(-\Box_g - V)\phi \tag{2.1}$$

where the covariant D'Alembert operator $\Box_g$ is defined as

$$\Box_g \phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu \left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu \phi\right)$$

2.1 Euclidean action

Performing an analytic continuation of the action to the Euclidean time $x \equiv (t, \mathbf{x}) \to \tilde{x} \equiv (\tau, \mathbf{x}) = (it, \mathbf{x})$ will imply that the determinant of the determinant of the new metric,

$$g_{\alpha\beta}(t, \mathbf{x}) = \tilde{g}_{\mu\nu}(\tilde{t}, \mathbf{x}) \frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial \tilde{x}^\nu}{\partial x^\beta},$$

changes its sign: $g \equiv \det g_{\mu\nu} = -\tilde{g}$, given that one row and one column of the new matrix are multiplied by i. Nevertheless, the covariant volume measure $\sqrt{-g}\ \mathrm{d}^{2\omega}x$ remains invariant under the previous transformation, as we expect from its definition. Finally, the scalar field also remains unchanged under this transformation, so we can write the Euclidean action (thus the superindex "(E)") as

$$S_E\left[\phi^{(E)}, g_{\mu\nu}^{(E)}\right] = \frac{1}{i} S[\phi, g_{\mu\nu}]_{t=-i\tau} = \frac{1}{2} \int \mathrm{d}^{2\omega} x^{(E)} \sqrt{g^{(E)}}\ \left[\phi^{(E)} (\Box_{g^{(E)}} + V) \phi^{(E)}\right]. \tag{2.2}$$

Lastly, we change the overall sign of the Euclidean metric, with signature (- - - -), and define a new metric $\gamma_{\mu\nu}(\tau, \mathbf{x}) \equiv -g_{\mu\nu}^{(E)}(\tau, \mathbf{x}) = -g_{\mu\nu}(t, \mathbf{x})|_{t=-i\tau}$. The determinant does not change under this last transformation, so $\gamma = g^{(E)}$; but the Laplace operator does (since it depends linearly on the metric), so $\Box_\gamma = -\Box_{g^{(E)}}$. Now that we have introduced the metric, in order to relief the notation, we will drop the superindex (E) and denote the Euclidean variables simply by $g_{\mu\nu}, \phi$ and x. With this notation the covariant D'Alembert operator becomes

$$\Box_g \phi = \frac{1}{\sqrt{g}} \partial_\mu \left[\sqrt{g}\, g^{\mu\nu} \partial_\nu \phi\right], \tag{2.3}$$

and the Euclidean action

$$S_E[\phi, g_{\mu\nu}] = \frac{1}{2} \int \mathrm{d}^{2\omega} x \sqrt{g} \left[\phi(x)(-\Box_g + V(x))\phi(x)\right] \tag{2.4}$$

2.2 Functional determinants

The following step to calculate the Euclidean effective action, as written in equation (1.4) (and considering the non-trivial action $g_{\mu\nu}$ as an external field), is to express it in terms of a functional derivative. To do so, we consider eigenvalues λ_n and eigenvectors ϕ_n of the operator entering the action:

$$[-\Box_g + V(x)]\,\phi_n(x) = \lambda_n\phi_n \tag{2.5}$$

These eigenvectors form a complete orthonormal basis in which we can expand the scalar field: $\phi(x) = \sum_n c_n\phi_n$ [1], and write the Euclidean action (2.4) as

$$S_E[\phi, g_{\mu\nu}] = \frac{1}{2}\int \mathrm{d}^{2\omega}x\,\sqrt{g}\sum_{n,m} c_m\phi_m\lambda_n c_n\phi_n = \frac{1}{2}\sum_n c_n^2\lambda_n$$

where we used the property of orthogonality, $\int \mathrm{d}^{2\omega}x\,\sqrt{g}\,\phi_m(x)\phi_n(x) = \delta_{mn}$.

By construction, both c_n and λ_n are generally covariant (i.e. coordinate independent), thus, if we conveniently define the path integral measure as $\mathcal{D}\phi = \prod_n(\mathrm{d}c_n\,/\sqrt{2\phi})$, then the path integral defining the Euclidean effective action reduces to

$$\int \mathcal{D}\phi\,\exp\left(-S_E[\phi, g_{\mu,\nu}]\right) = \int \prod_{n=0}^{\infty}\left\{\frac{\mathrm{d}c_n}{\sqrt{2\pi}}\,\exp\left(-\frac{1}{2}\lambda_n c_n^2\right)\right\} = \left[\prod_{n=0}^{\infty}\lambda_n\right]^{-1/2}$$

The product inside the brackets in the r.h.s. of this expression is simply the determinant[2] of the operator $[-\Box_g + V(x)]$, according to equation (2.5). Thus we can formally evaluate the Euclidean effective action in equation (1.4) as

$$\Gamma_E[g_{\mu\nu}] = -\log\left[\int \mathcal{D}q\,e^{-S_E[\phi, g_{\mu\nu}]}\right] = \frac{1}{2}\log\left[\prod_n \lambda_n\right] = \frac{1}{2}\log\det[-\Box_g + V] \tag{2.6}$$

Remark: Hilbert space versus space of functions

In the previous section we considered the eigenvectors and eigenvalues of a linear operator $-\Box_g + V(x)$ acting on a space of functions $\phi(x)$ (see equation (2.5)). Nevertheless, the actual operator acts on general vectors of the form $|\psi\}$ belonging to an auxiliary Hilbert space in which we can calculate the functional determinant. In such a Hilbert space, vectors are defined as

[1] with $c_n = \int \mathrm{d}^{2\omega}x\,\sqrt{g}\phi(x)\phi_n(x)$

[2] Although rigorously speaking, a generalization of the determinant for infinite-dimensional operators would be needed.

$$|\psi\rangle = \int \mathrm{d}^{2\omega}x\, \psi(x)|x\rangle,$$

with $\psi(x) = \langle x|\psi\rangle$ and $\langle x|x'\rangle = \delta(x - x')$, the Dirac delta function in 2ω dimensions. The scalar product of two *vectors* in this Hilbert space is defined by

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^{2\omega}x\, \psi_1(x)\psi_2(x),$$

which is manifestly non-covariant. On the other side, the inner product between two *functions* ϕ_1 and ϕ_2 follows the covariant inner product

$$(\phi_1, \phi_2) \equiv \int \mathrm{d}^{2\omega}x\, \sqrt{g}\, \phi_1(x)\phi_2(x).$$

It is clear that these inner products only differ by a factor $\sqrt{g}$. So a mapping $\psi(x) \leftrightarrow \phi(x)$ which preserves the scalar product, i.e. $(\phi_1, \phi_2) = \langle\psi_1|\psi_2\rangle$, must fulfil $\psi(x) = \langle x|\psi\rangle = g^{1/4}\phi(x)$. Furthermore, if we map the self-adjoint differential operator $-\Box_g + V$ to a Hermitian operator acting in the Hilbert space $\hat{M}$, then its representation by matrix elements in the $|x\rangle$ basis is:

$$\langle x|\hat{M}|x'\rangle = g^{1/4}(x)\left(-\Box_{g(x)} + V\right)\left[g^{-1/4}(x)\delta(x - x')\right]. \tag{2.7}$$

The subscript $g(x)$ in equation (2.7) emphasizes the fact that the derivatives involved in the operator $\Box$ are with respect to x rather than x'.

2.3 Zeta functions

Our differential operator $-\Box_g$ has increasingly growing eigenvalues so the determinant $\prod_n \lambda_n$ does not converge. We then need to renormalize it using the so-called *zeta function of the operator* $\hat{M}$, defined by

$$\zeta_M(s) \equiv \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n}\right)^s, \tag{2.8}$$

analogously to the definition of Riemann's ζ function. The previous sum converges for large enough values of s and can be analytically continued for all other smaller values. We can relate this function to the functional determinant in equation (2.6) by noting that

$$\left.\frac{\mathrm{d}\zeta_M(s)}{\mathrm{d}s}\right|_{s=0} = \left[-\sum_n e^{-s\log\lambda_n}\log\lambda_n\right]_{s=0} = -\log\prod_n \lambda_n = -\log\det\hat{M}. \tag{2.9}$$

This provides a definition of the determinant of an infinite-dimensional operator. However, this method also requires to know all eigenvalues of the operator, so we introduce, at last, the heat kernel as a tool to compute the ζ_M function.

2.4 Heat kernels

The heat kernel of the operator $\hat{M}$ is an operator $\hat{K}_M(\tau)$ depending on a scalar parameter τ (which will later be interpreted as the proper time) and the eigenvalues λ_n and orthonormal eigenvectors $|\psi_n\rangle$ of $\hat{M}$:

$$\hat{K}_M(\tau) \equiv \sum_n e^{-\lambda_n\tau}|\psi_n\rangle\langle\psi_n|. \tag{2.10}$$

One property of this operator is $\hat{K}_M(\tau = 0) = \hat{1}$. It is also clearly well defined for $\tau > 0$, and its trace takes up an easy expression in the basis $|\psi_n\rangle$:

$$\mathrm{Tr}\,\hat{K}_M(\tau) = \sum_n \langle\psi_n|\hat{K}_M(\tau)|\psi_n\rangle = \sum_n e^{-\lambda\tau}.$$

Now, we can use this trace, along with a rescaled version of Euler's gamma function to find an expression for $\zeta_M(s)$. We take $u = \lambda_n\tau$ and express

$$\Gamma(s) = \int_0^\infty e^{-u}u^{s-1}\,\mathrm{d}u = \lambda_n^s\int_0^\infty e^{-\lambda\tau}\tau^{s-1}\,\mathrm{d}\tau\,,$$

with $\mathrm{Re}\,s > 0$. Putting together the two previous lines we find

$$\zeta_M(s) = \sum_n \lambda_n^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}\,\hat{K}_M(\tau)\,\tau^{s-1}\,\mathrm{d}\tau\,. \tag{2.11}$$

The first derivative of the heat kernel constitutes its most attractive property in order to construct it:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{K}_M(\tau) = -\sum_n e^{-\lambda_n\tau}\lambda_n|\psi_n\rangle\langle\psi_n| = -\hat{M}\hat{K}_M. \tag{2.12}$$

Applying the boundary condition $\hat{K}_M(0) = \hat{1}$, the solution to equation (2.12) is simply $\hat{K}_M(\tau) = \exp\left(-\tau\hat{M}\right)$, and the trace (invariant under change of basis) entering equation (2.11) is

$$\operatorname{Tr}\hat{K} = \int \mathrm{d}^{2\omega}x\, \langle x|\exp\left(-\tau\hat{M}\right)|x\rangle.$$

Letting $\hat{M} \equiv -\Delta$, the Laplace operator, the differential equation (2.12) is the *heat equation* describing the propagation of heat in a homogeneous medium. Hence the name *heat kernel*.

3 Computation of the heat kernel

In this section we calculate in detail the trace of the heat kernel of the operator $\hat{M}$ defined in equation (2.7) as a perturbative series. This computation relies on the assumption that the potential is small, $|V| \ll 1$, and that space is only weakly curved, which we express by decomposing the metric $g_{\mu\nu}$, with Euclidean signature, into a sum of $\delta_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$:

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + h_{\mu\nu}(x), \quad g^{\mu\nu}(x) = \delta^{\mu\nu} + h^{\mu\nu}(x). \tag{3.1}$$

The condition $g^{\mu\alpha}g_{\alpha\nu} = \delta^{\mu}_{\nu}$ implies that $h^{\mu\nu} = -h_{\alpha\beta}\delta^{\mu\alpha}\delta^{\nu\beta} + O^2(h_{\alpha\beta})$. This decomposition is not generally covariant since $\delta_{\mu\nu}$ is a fixed matrix, and the coordinate system must be taken so that $h_{\mu\nu}(x)$ is everywhere small, which is always possible in the assumption that space is almost flat.

The heat kernel can therefore be expressed as a perturbative expansion $\hat{K}_M(\tau) = \hat{K}_0(\tau) + \hat{K}_1(\tau) + \hat{K}_2(\tau) + \dots$, where $\hat{K}_0(\tau)$ is the heat kernel for strictly flat space (i.e. $g^{(0)}_{\mu\nu} = \delta_{\mu\nu}$) and $V = 0$; the following terms are progressively smaller corrections in $h_{\mu\nu}$ and V. We will calculate only up to the leading-order correction $\hat{K}_1(\tau)$.

3.1 Initial approximation to the heat kernel

According to equation (2.12), the flat space solution of the heat kernel $\hat{K}_0(\tau)$ obeys the differential equation

$$\frac{\mathrm{d}\hat{K}_0}{\mathrm{d}\tau} = \Box\hat{K}_0, \quad \hat{K}_0(0) = \hat{1}, \tag{3.2}$$

where $\Box$ is the regular Laplace operator in flat space. Then, $\hat{K}_0(\tau) = e^{\tau\Box}$. Now, to find an explicit solution for the matrix element $\langle x|\hat{K}_0(\tau)|y\rangle$, we expand the exponential $e^{\tau\Box}$ in power series series and use the Fourier representation of the Dirac delta function in 2ω dimensions $\delta(x-y) = \frac{1}{(2\pi)^{2\omega}}\int \mathrm{d}^{2\omega}k\, e^{ik(x-y)}$:

$$\langle x|\hat{K}_0(\tau)|y\rangle = e^{\tau\Box_x}\delta(x-y) = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\left[\sum_{n=0}^{\infty}\frac{(\tau\Box_x)^n}{n!}\right]e^{ik(x-y)} =$$
$$= \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}\left[\sum_{n}\frac{(-\tau k^2)^n}{n!}\right]e^{ik(x-y)} = \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^{2\omega}}e^{-\tau k^2+ik(x-y)}.$$

The expression is a Gaussian integral, once we reorder terms and complete the perfect square we get

$$\langle x|\hat{K}_0(\tau)|y\rangle = \frac{1}{(4\pi\tau)^{\omega}}\exp\left[-\frac{(x-y)^2}{4\tau}\right]. \tag{3.3}$$

Equation (3.3) is also the Green's function of the heat kernel equation (3.2).

3.2 Correction to the operator $\Box$

As we have seen, we defined the initial approximation to the heat kernel, $\hat{K}_0(\tau)$, as the operator satisfying the differential equation (3.3), which involves the Laplace operator $\Box_x$ in flat space. The complete heat kernel $\hat{K}_M(\tau)$, satisfies equation (2.12), which involves the more general operator $\hat{M}$, defined in equation (2.7):

$$\langle x|\hat{M}|x'\rangle = g^{1/4}(x)\left(-\Box_{g(x)} + V\right)\left[g^{-1/4}(x)\delta(x-x')\right]$$
$$= -g^{1/4}\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^{\nu}}\left[g^{\mu\nu}\sqrt{g}\frac{\partial}{\partial x^{\mu}}\left(g^{-1/4}\delta(x-x')\right)\right] + V\,\delta(x-x').$$

It is convenient (for reasons we will shortly see) to express this operator as a sum of the flat-space Laplace operator $\Box$ plus a small correction $\hat{s}$ depending on V and $h_{\mu\nu}$: $-\hat{M} = \Box + \hat{s}$. Now, to simplify the calculations, we split the correction $\hat{s}$ into a sum of three terms: $\hat{s} = \hat{h} + \hat{\Gamma} + \hat{P}$. The explicit derivation of these terms is somehow cumbersome but straightforward, and makes use of the identity $(\log g)_{,\mu} = g^{\alpha\beta}g_{\alpha\beta,\mu}$. Omitting the potential term and the arguments in the Dirac delta we have:

$$
\begin{aligned}
g^{-1/4}\partial_\nu\left[g^{\mu\nu}\sqrt{g}\,\partial_\mu\left(g^{-1/4}\delta\right)\right] &= \\
= g^{+1/4}g^{\mu\nu}\left(g^{-1/4}\delta\right)_{,\mu\nu} + g^{-1/4}\left(g^{\mu\nu}\sqrt{g}\right)_{,\nu}\left(g^{-1/4}\delta\right)_{,\mu} &= \\
= g^{\mu\nu}\left(\delta_{,\mu} - \frac{1}{4}(\log g)_{,\mu}\,\delta\right)_{,\nu} + \left(g^{\mu\nu}_{,\nu} + \frac{1}{4}g^{\mu\nu}(\log g)_{,\nu}\right)\left(\delta_{,\mu} - \frac{1}{4}(\log g)_{,\mu}\,\delta\right)&.
\end{aligned}
$$

The term containing the second derivative of δ is

$$
g^{\mu\nu}\delta(x-x') = (\delta^{\mu\nu} + h^{\mu\nu})\partial_\mu\partial_\nu\delta(x-x') = \langle x|\Box + \hat{h}|x'\rangle, \qquad (3.4)
$$

defining the operator $\hat{h}$ and the well-known flat-space Laplace operator $\Box$. Two of the terms involving a first derivative of δ cancel out and we are left with a simple expression defining $\hat{\Gamma}$:

$$
\frac{1}{4}g^{\mu\nu}\left[(\log g)_{,\nu}\delta_{,\mu} - (\log g)_{,\mu}\delta_{,\nu}\right] + g^{\mu\nu}_{,\nu}\delta_{,\mu} = h^{\mu\nu}_{,\mu}\partial_\nu\delta(x-x') = \langle x|\hat{\Gamma}|x'\rangle, \quad (3.5)
$$

where we have interchanged the dummy indices μ and ν and used the symmetry of $g^{\mu\nu}$ to cancel the two first terms. Also $g^{\mu\nu}_{,\nu} = h^{\mu\nu}_{,\nu}$. Finally, the remaining terms define the operator $\langle x|\hat{P}|x\rangle = P(x)\delta(x-x')$:

$$
\begin{aligned}
P(x) + V(x) = -\frac{1}{4}g^{\mu\nu}(\log g)_{,\mu\nu} - \frac{1}{4}\left(g^{\mu\nu}_{,\nu} + \frac{1}{4}g^{\mu\nu}(\log g)_{,\nu}\right)(\log g)_{,\mu} &= \\
= -\frac{1}{4}g^{\mu\nu}\left(g^{\alpha\beta}h_{\alpha\beta,\mu\nu} + h^{\alpha\beta}_{,\mu}h_{\alpha\beta,\nu} + h_{\mu\nu,\alpha}h^{\alpha\beta}_{,\beta} + \frac{1}{4}g^{\alpha\beta}g^{\kappa\lambda}h_{\alpha\beta,\mu}h_{\kappa\lambda,\nu}\right)& \\
\approx -\frac{1}{4}\delta_{\mu\nu}\Box h^{\mu\nu}(x) + O(h^2)&.
\end{aligned}
$$

We will make use of the last approximation later on.

3.3 Solving the differential equation for $\hat{K}_1(\tau)$

In the previous section we detailed the expression $-\hat{M} = \Box + \hat{s}$. With this decomposition, we can rewrite equation (2.12) in the following way:

$$
\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{K}_M = (\Box + \hat{s})\hat{K}_M, \qquad \hat{K}_M(0) = \hat{1}.
$$

If we now plug $\hat{K}_M = \hat{K}_0 + \hat{K}_1$ and use equation (3.2) to get rid of $\mathrm{d}\hat{K}_0 \big/ \mathrm{d}\tau$, we are left with the differential equation for $\hat{K}_1(\tau)$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{K}_1 = (\Box + \hat{s})\hat{K}_1 + \hat{s}\hat{K}_0 \approx \Box\hat{K}_1 + \hat{s}\hat{K}_0, \qquad \hat{K}_1(0) = \hat{1}, \tag{3.6}$$

where in the last step we neglected the second-order term $\hat{s}\hat{K}_1$. Given that $\hat{K}_0$ is already known, we make the following ansatz: $\hat{K}_1(\tau) = \hat{K}_0\hat{C}(\tau)$ with an undetermined auxiliary function $\hat{C}(\tau)$ which we can solve by plugging this ansatz into equation (3.6):

$$\hat{K}_0(\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}\hat{C}(\tau) = \hat{s}\hat{K}_0(\tau) \Rightarrow \hat{C}(\tau) = \int_0^\tau \mathrm{d}\tau'\, \hat{K}_0^{-1}(\tau')\hat{s}\hat{K}_0(\tau').$$

Equation (3.2) involves only the linear Laplace operator, which means that $\hat{K}_0(\tau)\hat{K}_0(\tau') = \hat{K}_0(\tau+\tau')$. And this also implies $\hat{K}_0^{-1}(\tau) = \hat{K}_0(-\tau)$. Therefore we can write

$$\hat{K}_1(\tau) = \hat{K}_0(\tau)\hat{C}(\tau) = \int_0^\tau \mathrm{d}\tau'\, \hat{K}_0(\tau - \tau')\hat{s}\hat{K}_0(\tau'). \tag{3.7}$$

Since equation (3.7) is linear in $\hat{s} = \hat{h} + \hat{\Gamma} + \hat{P}$, we can decompose $\hat{K}_1 = \hat{K}_1^h + \hat{K}_1^\Gamma + \hat{K}_1^P$. We shall first calculate $\hat{K}_1^P$ since the other two terms are easily computed using this result.

3.4 Computation of the matrix elements of $\hat{K}_1^P$

The matrix elements of interest to evaluate the trace of $\hat{K}_1^P(\tau)$ are the diagonal ones, i.e. $\langle x|\hat{K}_1^P|x\rangle$. However, these can also be simply obtained from the nondiagonal terms $\langle x|\hat{K}_1^P|y\rangle$. For this calculation we will make use of the definition of an orthonormal basis in Hilbert space: $\hat{1} = \int \mathrm{d}^{2\omega}x\, |x\rangle\langle x|$. From equation (3.7) we have:

$$\begin{aligned}
\langle x|\hat{K}_1^P|y\rangle &= \int_0^\tau \mathrm{d}\tau'\, \langle x|\hat{K}_0(\tau-\tau')\hat{P}\hat{K}_0(\tau')|y\rangle = \\
= \int_0^\tau \mathrm{d}\tau'\, \langle x|\hat{K}_0(\tau-\tau')&\int \mathrm{d}^{2\omega}z\, |z\rangle\langle z|\hat{P}\int \mathrm{d}^{2\omega}w\, |w\rangle\langle w|\hat{K}_0(\tau')|y\rangle = \\
= \int_0^\tau \mathrm{d}\tau' &\int \mathrm{d}^{2\omega}z\, \langle x|\hat{K}_0(\tau-\tau')|z\rangle P(z)\langle z|\hat{K}_0(\tau')|y\rangle = \\
= \int_0^\tau \int \mathrm{d}^{2\omega}z\, &\frac{1}{[4\pi(\tau-\tau')4\pi\tau']^\omega}\exp\left[-\frac{(x-z)^2}{4(\tau-\tau')} - \frac{(z-y)^2}{4\tau'}\right] P(z),
\end{aligned}$$

where, in the second line, we have used $\langle z|\hat{P}|w\rangle = P(z)\delta(z-w)$ and integrated over w to get rid of the Dirac delta; and in the last line we have made use of the result in equation (3.3) for $\langle x|\hat{K}_0(\tau)|y\rangle$. To go on from this point we introduce the Fourier transform of $P(z) = \frac{1}{(2\pi)^\omega}\int \mathrm{d}^{2\omega}k\, e^{ikz}p(k)$:

$$\langle x|\hat{K}_1^P|y\rangle = \int_0^\tau \int \mathrm{d}^{2\omega}z\, \frac{\mathrm{d}^{2\omega}k}{(2\pi)^\omega} \frac{\exp\left[-\frac{(x-z)^2}{4(\tau-\tau')} - \frac{(z-y)^2}{4\tau'} + ikz\right]}{[4\pi(\tau-\tau')4\pi\tau']^\omega}\, p(k) =$$

$$= \frac{\exp\left[-\frac{(x-y)^2}{4\tau}\right]}{(4\pi\tau)^\omega} \int_0^\tau \int \frac{\mathrm{d}^{2\omega}k}{(2\pi)^\omega} \exp\left[-\frac{\tau'(\tau-\tau')k^2}{\tau} + \frac{ik}{\tau}\left(x\tau' + y(\tau-\tau')\right)\right] p(k),$$

where we have completed the perfect square to perform the Gaussian integral over z. By taking the limit $y \to x$ we recover the diagonal term, which, after integration over k results in

$$\langle x|\hat{K}_1^P|x\rangle = \frac{1}{(4\pi\tau)^\omega}\int_0^\tau \mathrm{d}\tau \exp\left[\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right] P(x) \qquad (3.8)$$

3.5 Computation of $\hat{K}_1^\Gamma$ and $\hat{K}_1^h$

The matrix elements of the operators $\hat{\Gamma}$ and $\hat{h}$ contain one and two derivatives of the Dirac delta function, respectively, unlike $\hat{P}$, which contained none. This means that the computation procedure must be modified. We will detail the calculation for $\hat{K}_1^\Gamma$ for the sake of simplicity. $\hat{K}_1^h$ follows an analog procedure.

$$\langle x|\hat{K}_1^\Gamma|y\rangle = \int_0^\tau \mathrm{d}\tau'\, \langle x|\hat{K}_0(\tau-\tau')\hat{\Gamma}\hat{K}_0(\tau')|y\rangle =$$

$$= \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}^{2\omega}z\, \langle x|\hat{K}_0(\tau-\tau')|z\rangle h^{\mu\nu}_{,\nu}(z)\frac{\partial}{\partial z^\mu}\langle z|\hat{K}_0(\tau')|y\rangle =$$

$$= -\frac{\partial}{\partial y^\mu}\int_0^\tau \mathrm{d}\tau' \int \mathrm{d}^{2\omega}z\, \langle x|\hat{K}_0(\tau-\tau')|z\rangle h^{\mu\nu}_{,\nu}(z)\langle z|\hat{K}_0(\tau')|y\rangle =$$

$$.$$

In the last line we substituted the derivative ∂_z by ∂_y by noting that $\langle z|\hat{K}_0(\tau)|y\rangle$ (the only term the derivative acts on), is a function only of $(z-y)$ in the exponential. Now this result can be treated analogous to $\langle x|\hat{K}_1^P|y\rangle$ by substituting $P(z)$ by $h^{\mu\nu}_{,\nu}(z)$. So we have

$$\langle x|\hat{K}_1^{\Gamma}|x\rangle = -\lim_{y\to x}\frac{\partial}{\partial y^{\mu}}\,\langle x|\hat{K}_1^{P}|y\rangle\Big|_{P(z)\to h^{\mu\nu}_{,\nu}(z)} =$$
$$= -\frac{1}{(4\pi\tau)^{\omega}}\int_0^{\tau}\mathrm{d}\tau'\exp\left[\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right]\frac{\tau-\tau'}{\tau}h^{\mu\nu}_{,\mu\nu}(x).$$

As mentioned above, the computation for the matrix element of $\hat{K}_1^h$ is similar so we only sketch here one of the steps required and the final result:

$$\langle x|\hat{K}_1^{h}|x\rangle = -\lim_{y\to x}\frac{\partial}{\partial y^{\nu}}\frac{\partial}{\partial y^{\mu}}\,\langle x|\hat{K}_1^{P}|y\rangle\Big|_{P(z)\to h^{\mu\nu}(z)} =$$
$$= -\frac{1}{(4\pi\tau)^{\omega}}\int_0^{\tau}\mathrm{d}\tau'\exp\left[\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right]\left\{-\frac{\delta_{\mu\nu}}{2\tau}+\left(\frac{\tau-\tau'}{\tau}\right)^2\partial_\mu\partial_\nu\right\}h^{\mu\nu}(x).$$

3.6 Trace of the heat kernel

To conclude we summarize all the previous results to compute the trace of the heat kernel, $\mathrm{Tr}\,\hat{K}$. We start by expanding in series the exponential

$$\exp\left[\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right]\approx\hat{1}+\frac{\tau'(\tau-\tau')}{\tau}\Box_x,$$

up to leading order. We also neglect terms of order h^2 in $P(x)$, and approximate (as already anticipated) $P(x)\approx-\frac{1}{4}\delta_{\mu\nu}\Box h^{\mu\nu}(x)-V(x)+O(h^2)$. Then we have:

$$\langle x|\hat{K}_1(\tau)|x\rangle = \langle x|\hat{K}_1^{P}+\hat{K}_1^{h}+\hat{K}_1^{\Gamma}|x\rangle =$$
$$= \frac{1}{(4\pi\tau)^{\tau}}\int_0^{\tau}\mathrm{d}\tau'\left[\hat{1}+\frac{\tau'(\tau-\tau')}{\tau}\Box_x\right]\left\{P(x)-\frac{\delta_{\mu\nu}}{2\tau}-\frac{\tau'(\tau-\tau')}{\tau^2}\partial_\mu\partial_\nu\right\}h^{\mu\nu}(x) =$$
$$= \frac{1}{(4\pi\tau)^{\omega}}\left\{P(x)\tau-\frac{1}{2}\delta_{\mu\nu}h^{\mu\nu}-\frac{1}{6}\tau h^{\mu\nu}_{,\mu\nu}+\frac{\tau}{6}\Box P-\frac{\tau}{12}\delta_{\mu\nu}\Box h^{\mu\nu}-\frac{\tau}{30}\Box h^{\mu\nu}_{,\mu\nu}\right\} =$$
$$= \frac{1}{(4\pi\tau)^{\omega}}\left\{-\frac{1}{2}\delta_{\mu\nu}h^{\mu\nu}-V(x)\tau+\frac{\tau}{6}[\delta_{\mu\nu}\Box h^{\mu\nu}-h^{\mu\nu}_{,\mu\nu}]+\Box(\dots)\right\},$$

where the omitted terms (under the last dots) involve fourth derivatives of $h^{\mu\nu}$. This expression can be rewritten in a manifestly covariant form by noting that, for the metric (3.1), the volume factor $\sqrt{g}$ and the Ricci scalar R are:

$$\sqrt{g} = 1 - \frac{1}{2}\delta_{\mu\nu}h^{\mu\nu} + O(h^2), \qquad R = \delta_{\mu\nu}\Box h^{\mu\nu} - h^{\mu\nu}_{,\mu\nu} + O(h^2).$$

Therefore, summing up all results, we get

$$\operatorname{Tr}\hat{K}(\tau) = \int \mathrm{d}^{2\omega}x \,\langle x|\hat{K}_0 + \hat{K}_1|x\rangle = \int \frac{\mathrm{d}^{2\omega}x\,\sqrt{g}}{(4\pi\tau)^{\omega}}\left[1 + \left(\frac{R}{6} - V\right)\tau + O(h^2)\right]. \tag{3.9}$$

To conclude, it is worth commenting that it is possible to perform higher-order approximation for $\operatorname{Tr}\hat{K}$ involving higher-order terms such as R^2, V^2, VR and $R_{\mu\nu}R^{\mu\nu}$ (for the second-order one). The next natural step is to calculate the effective action using equations (2.9) and (2.11) involving the zeta functions of the operator $\hat{M}$; and to study the ultraviolet divergences when $\tau \to 0$. A renormalization procedure is required then but it remains out of the scope of this paper.