The Schwinger Effect

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This is a review of the Schwinger effect in quantum electrodynamics. It results in the Schwinger formula, which describes the probability for pair creation out of the vacuum in the presence of a constant electric field only. These considerations will take place in the framework of scalar QED, as well as spinor QED.

1 Introduction

This report is a brief introduction to the Schwinger Effect of relativistic quantum electrodynamics. This is a semi-classical description of particle-antiparticle creation in the vacuum caused by the presence of strong classical electric fields. There are various reasons for which it is rewarding to study the Schwinger Effect. First of all, it will turn out to be a non-perturbative prediction of quantum electrodynamics. This enables us to study and test QED in a regime which is contrary to the perturbative approach of Feynman diagrams often taken in lectures. Whilst the latter has produced predictions which have been experimentally verified to highest precision, the Schwinger effect has not been observed to present date. Therefore, it is ongoing research to develop appropriate experimental set-ups.

Secondly, the equations describing the Schwinger effect in QED are those of a quantum field with a time dependent mass and can be applied to problems in related fields like cosmology or quantum chromodynamics. Particle creation due to a rapidly expanding universe during the period of inflation, for instance, differs from the Schwinger effect by no more than a canonical transformation [Mar07]. Applying this mechanism to besaid cosmological case, eventually allows to test our model of inflation. Also, we can learn about QCD and quark production in strong gluon fields of heavy ion collisions. [AHR$^+$01]

Lastly, we will find a critical scale for the electric field strength in order to obtain pair creation. This so-called Schwinger limit describes a transition into a regime, where the back reaction of quantum effects on the classical electromagnetic fields can no longer be neglected. This gives rise to non-linear effects in classical electrodynamics such as light-light interaction. Early works by Heisenberg and Euler have shown how to obtain an effective action describing these effects. [HE36]

This report will closely follow the approach of [GMM94] resulting in the Schwinger formula, which has been derived differently by Julian Schwinger in 1951 [Sch51].

2 The Schwinger Effect for Complex Scalar Fields

Let us start our considerations with the action S of a complex scalar field ϕ of mass m in a four dimensional Minkowski spacetime:

$$S = \int \mathrm{d}^4x \left(\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi \phi^*\right) \tag{1}$$

Also, let the fields be defined on a finite but large volume V which is treated in the continuum limit. The action above is invariant under a *global* $U(1)$ phase rotation of the fields ϕ, ϕ^*. If we promote this global symmetry to a *local* one, the partial derivatives need to be replaced by *covariant* derivatives:

$$\partial_\mu \longrightarrow \nabla_\mu = \partial_\mu + iqA_\mu \ , \tag{2}$$

where q is an electric charge and A_μ is the *gauge* field associated with the four-vector potential known from classical electrodynamics. In order to obtain the full quantum theory one would usually proceed by adding a term to the Lagrangian which specifies the dynamics of our gauge field. However, we want to describe the fields' interactions with a classical source. That means the gauge field will not be quantised, as opposed to the scalar one, but we will assume it to be given externally:

$$A^\mu = \left(0, 0, 0, A^3(t)\right) \ . \tag{3}$$

With this, the electric field strength reads:

$$\vec{E} = \left(0, 0, -\frac{\mathrm{d}A^3}{\mathrm{d}t}\right) \ . \tag{4}$$

For simplicity reasons, the field is assumed to be spatially constant. Furthermore, we require the electric field strength to be switched off adiabatically at $t \to \pm\infty$, which means that in these limits A^3 approaches constant functions $A_\pm$:

$$\lim_{t \to \pm\infty} A^3(t) = A_\pm \ . \tag{5}$$

Expanding the covariant derivatives, the action turns into the following:

$$S = \int \mathrm{d}^4x \left(\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi\phi^* + iq(\partial_\mu \phi)\phi^* A^\mu - iq(\partial_\mu \phi^*)\phi A^\mu + q^2 A^\mu A_\mu \phi\phi^*\right) \ . \tag{6}$$

The equation of motion for ϕ is obtained from the variational principle:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0 \ , \tag{7}$$

while the one for ϕ^* can be obtained likewise. Subsequently, this yields:

$$\left(\partial^2 + m^2 - 2iqA^3\partial_3 + q^2 A_3^2\right)\phi = 0 \ . \tag{8}$$

The equation of motion can be translated to Fourier space, and then reads:

$$\ddot{\phi}_k + \omega_k^2(t)\phi_k = 0 \ , \tag{9}$$

$$w_k^2(t) = m^2 + \underbrace{k_1^2 + k_2^2}_{=:k_\perp^2} + (k_3 - qA_3)^2 \ , \tag{10}$$

From equation (5), one finds that ω loses its time dependence in the limit of late times:

$$\lim_{t\to\pm\infty} \omega(\vec{k}, t) = \omega_\pm(\vec{k}) \ . \tag{11}$$

It is noted, that the above equation is that of a *parametric* oscillator, the harmonic oscillator with a time-dependent frequency. Solving this partial differential equation will be one of the main tasks of dealing with the Schwinger effect within different fields of particle physics. For now, assume the above equation (9) is solved by the following mode functions:

$$\ddot{f}^\pm(\vec{k}, t) + \omega_k^2(t) f^\pm(\vec{k}, t) = 0 \ . \tag{12}$$

Then a general solution of equation (9) can be written as an expansion in mode functions $f^\pm$, where x denotes the position in spacetime $x := (t, \vec{x})$:

$$\phi(x) = \int \mathrm{d}^3k \left(\phi^-(\vec{k}, x)\alpha^-(\vec{k}) + \phi^+(-\vec{k}, x)\beta^+(\vec{k}) \right) \ , \tag{13}$$

$$\phi^*(x) = \int \mathrm{d}^3k \left(\phi^{*^-}(\vec{k}, x)\alpha^+(\vec{k}) + \phi^{*^+}(-\vec{k}, x)\beta^-(\vec{k}) \right) \ , \tag{14}$$

$$\phi^\pm(\vec{k}, t) = \frac{1}{\sqrt{(2\pi)^3 2\omega_-}} \ e^{i\vec{k}\vec{x}} f^\pm(\vec{k}, t) \ , \tag{15}$$

where $\alpha^- = (\alpha^+)^*$ and $\beta^- = (\beta^+)^*$. In some cases it is possible to have the mode functions and their associated coefficients chosen such that they only depend on the modulus of $\vec{k}$. For example, such a choice is possible when the Schwinger effect is applied to cosmological particle production because of the isotropy of spacetime there. [MW07]
However, in our case this symmetry is broken due to the presence of the electric field along $3-$direction. We keep this in mind and drop the vector dependence in our notation from now on.
We may now quantise the complex scalar field by promoting the coefficients α, β to quantum mechanical ladder operators

$$\alpha_p^- \longrightarrow a_p \ , \quad \alpha_p^+ \longrightarrow a_p^\dagger \ , \quad \beta_p^- \longrightarrow b_p \ , \quad \beta_p^+ \longrightarrow b_p^\dagger \ , \tag{16}$$

and subsequently requiring the following commutation relations:

$$\left[\beta_k, \beta_p^\dagger\right] = \left[\alpha_k, \alpha_p^\dagger\right] = \delta_{k,p} \overset{V\to\infty}{\longrightarrow} (2\pi)^{3/2}\delta_D(p-k) \ , \tag{17}$$

which involve Dirac delta functions in the continuum limit of a large volume V. The different units are accounted by a change in the operators' normalisations. All other commutators vanish.
The Hamiltonian $\mathcal{H}$ can be obtained from the Lagrangian density $\mathcal{L}$ as follows:

$$\mathcal{H} = \int \mathrm{d}^3x \left(\dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} + \dot{\phi}^*\frac{\partial\mathcal{L}}{\partial\dot{\phi}^*} - \mathcal{L}(\phi, \phi^*) \right) \tag{18}$$

Plugging in the mode expansions (13) and (14), as well as the commutation relations (17) yields:

$$\mathcal{H} = \int \mathrm{d}^3p \left(E(t,p) \left[a_p^\dagger a_p + b_{-p} b_{-p}^\dagger \right] + F^*(t,p)\, a_p b_{-p} + F(t,p)\, a_p^\dagger b_{-p}^\dagger \right) \tag{19}$$

In the above, the functions E and F have been defined as:

$$E(t,p) = \frac{1}{2\omega_-(p)\omega_p(t)} \left(\left| \dot{f}^+(t,p) \right|^2 + \left| f^+(t,p) \right|^2 \right) \quad , \tag{20}$$

$$F(t,p) = \frac{1}{2\omega_-(p)\omega_p(t)} \left(\dot{f}^+(t,p)^2 + f^+(t,p)^2 \right) . \tag{21}$$

In (18) the Hamiltonian contains a diagonal part ("$a_p^\dagger a_p + b_{-p} b_{-p}^\dagger$"), as well as an interaction part ("$a_p b_{-p}$" and "$a_p^\dagger b_{-p}^\dagger$"), which accounts for annihilation and creation of particle-antiparticle pairs with zero total momentum. Initially, at $t_{-\infty}$ ($t \to -\infty$), the external field is switched off and the Hamiltonian is diagonalised by the following mode functions, which are solutions to (12) and well-known from standard scalar quantum field theory:

$$f^\pm(t,p) = e^{\pm i\omega_-(p)t} . \tag{22}$$

In the following the *Bogolyubov* transformation of our ladder operators a_p and b_p at $t_{-\infty}$ into a new basis of creation and annihilation operators is introduced:

$$\begin{pmatrix} a_p \\ b_{-p}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_p^*(t) & -\beta_p(t) \\ -\beta_p^*(t) & \alpha_p(t) \end{pmatrix}}_{=:V_p} \begin{pmatrix} c_p \\ d_{-p}^\dagger \end{pmatrix} . \tag{23}$$

Initially, we want to recover our initial operators, hence $\alpha_p(t_{-\infty}) = 1$ and $\beta_p(t_{-\infty}) = 0$. If we require that the above transformation is invertible and commutation relations to be still valid for our new operators, we may conclude:

$$\det(V_p) = |\alpha_p(t)|^2 - |\beta_p(t)|^2 = 1 . \tag{24}$$

Choosing the respective coefficients in a specific manner:

$$|\beta_p(t)|^2 = \frac{E(t,p) - 1}{2} \quad , \quad \frac{\beta_p(t)}{\alpha_p(t)} = \frac{E(t,p) - 1}{F^*(t,p)} , \tag{25}$$

enables us to diagonalise the Hamiltonian $\mathcal{H}$ and to find its minimal energy eigenstate:

$$\mathcal{H} = \int \mathrm{d}^3p \; 2\omega(t,p) \left[c_p^\dagger c_p + d_{-p} d_{-p}^\dagger \right] . \tag{26}$$

The eigenstate with minimal energy eigenvalue, i.e. the ground state, is the one which is annihilated by both c_p and d_p and is called the *vacuum*. Note that this is a choice which is only valid at a certain instant in time t since the Bogolyubov coefficients and their associated quantum operators are dependant on time. Hence, the vacuum we are looking for is the instantaneous vacuum at time t:

$$c_p \left|0_t\right\rangle = d_p \left|0_t\right\rangle = 0 \quad \forall p \tag{27}$$

Fock space is now constructed by the subsequent application of creation operators to the vacuum in the usual fashion. As stated above, the vacuum changes with time and with it does the notion of particles. Two vacuum states at two different times $t_{-\infty}$ and t are related by the formula:

$$|0_t\rangle = \prod_p \frac{1}{|\alpha_p(t)|} \exp\left(-\frac{\beta_p}{\alpha_p} b^\dagger_{-p} a^\dagger_p \right) |0_{\rm in}\rangle \ , \tag{28}$$

where the different momentum signs in the creation operators of the exponent indicate that particles and antiparticles are created with opposite momenta. This relation can be proven by checking explicitly that $|0_t\rangle$ is annihilated by the instantaneous annihilation operators. These operators can be written as:

$$c_p = \alpha_p a_p + \beta_p b^\dagger_{-p} \quad , \quad d_p = \beta_{-p} a^\dagger_{-p} + \alpha_{-p} b_p \ , \tag{29}$$

where we inverted the Bogolyubov transformation (23). Note the following:

$$a_p (a^\dagger_p)^n = a_p a^\dagger_p (a^\dagger_p)^{n-1} = (a^\dagger_p a_p + 1)(a^\dagger_p)^{n-1} = a^\dagger_p a_p (a^\dagger_p)^{n-1} + (a^\dagger_p)^{n-1} \tag{30}$$

This procedure can be repeated $(n$-1)-times to obtain:

$$a_p (a^\dagger_p)^n = (a^\dagger_p)^n a_p + n (a^\dagger_p)^{n-1} \ . \tag{31}$$

Using this, one may calculate:

$$\alpha_p a_p |0_t\rangle = \alpha_p a_p \prod_q \frac{1}{|\alpha_q|} \exp\left(-\frac{\beta_q}{\alpha_q} b^\dagger_{-q} a^\dagger_q \right) |0_{\rm in}\rangle = a_p \frac{\alpha_p}{|\alpha_p|} \exp\left(-\frac{\beta_p}{\alpha_p} b^\dagger_{-p} a^\dagger_p \right) |0_{\rm in}\rangle \tag{32}$$

$$= \frac{\alpha_p}{|\alpha_p|} \sum_{n=0}^\infty \frac{(-\beta_p/\alpha_p)^n}{n!} (b^\dagger_{-p})^n \left((a^\dagger_p)^n a_p + n (a^\dagger_p)^{n-1} \right) |0_{\rm in}\rangle \tag{33}$$

$$= \frac{\alpha_p}{|\alpha_p|} \frac{-\beta_p}{\alpha_p} b^\dagger_{-p} \sum_{n=1}^\infty \frac{(-\beta_p/\alpha_p)^{(n-1)}}{(n-1)!} (b^\dagger_{-p})^{n-1} (a^\dagger_p)^{n-1} |0_{\rm in}\rangle = -\beta_p b^\dagger_{-p} |0_t\rangle \tag{34}$$

Hence, the vacuum is indeed annihilated by its associated annihilation operator c_p:

$$c_p |0_t\rangle = 0 \quad \forall p \ . \tag{35}$$

Analogously, one can easily obtain the same result for d_p. Looking at equations (32) to (34), one may notice that a_p can be interpreted as a derivative with respect to $a^\dagger_p$ [MW07]. In fact one can consider Fock space in its holomorphic representation, where the following identifications can be made:

$$a^\dagger_p \longrightarrow \bar{z}_p \ , \ a_p \longrightarrow \frac{\partial}{\partial \bar{z}_p} \ , \ b^\dagger_p \longrightarrow \bar{y}_p \ , \ b_p \longrightarrow \frac{\partial}{\partial \bar{y}_p} \ , \tag{36}$$

with z_p and y_p being complex numbers and $\bar{z}_p$ and $\bar{y}_p$ their complex conjugates. The a_p and b_p vacuum is defined as:

$$f_{|0_{\rm in}\rangle} = 1 \ , \ \text{since} \ \frac{\partial}{\partial \bar{z}_p} f_{|0_{\rm in}\rangle} = \frac{\partial}{\partial \bar{y}_p} f_{|0_{\rm in}\rangle} = 0 \ . \tag{37}$$

In this representation one immediately notices that the anti-holomorphic exponential function

$$f(\bar{z}_p, \bar{y}_p) = \prod_p \frac{1}{|\alpha_p|} \exp\left(-\frac{\beta_p}{\alpha_p} \, \bar{y}_{-p} \bar{z}_p \right) \underbrace{f_{|0_{\rm in}\rangle}}_{=1} \tag{38}$$

is a solution to the following two partial differential equations:

$$0 = \left(\alpha_p \frac{\partial}{\partial \bar{z}_p} + \beta_p \bar{y}_{-p} \right) f(\bar{z}_p, \bar{y}_p) \tag{39}$$

$$0 = \left(\beta_{-p} \bar{z}_{-p} + \alpha_{-p} \frac{\partial}{\partial \bar{y}_p} \right) f(\bar{z}_p, \bar{y}_p) \ , \tag{40}$$

which is the analogue of the condition in (35) for c_p and d_p, respectively.
However, we also need to check that the vacuum preserves its normalisation. In order to do this, let us first extend relation (31) to:

$$(a_p)^n (a_p^\dagger)^n |0_{\rm in}\rangle = (a_p)^{n-1} a_p (a_p^\dagger)^n |0_{\rm in}\rangle \overset{(31)}{=} (a_p)^{n-1} n (a_p^\dagger)^{n-1} |0_{\rm in}\rangle = n! \, |0_{\rm in}\rangle \ , \tag{41}$$

or accordingly:

$$\left(\frac{\partial}{\partial \bar{z}_p} \right)^n (\bar{z}_p)^n = n! \ . \tag{42}$$

The norm of the vacuum can then be calculated as follows:

$$\langle 0_t | 0_t \rangle = \prod_p \prod_q \frac{1}{|\alpha_p||\alpha_q|} \sum_{n,m} \frac{1}{n!m!} \left(\frac{-\beta_q^*}{\alpha_q^*} \right)^m \left(\frac{-\beta_p}{\alpha_p} \right)^n \langle 0_{\rm in}| \, (a_q)^m (b_{-q})^m (b_{-p}^\dagger)^n (a_p^\dagger)^n |0_{\rm in}\rangle \ . \tag{43}$$

The only non-zero terms are those, where the number of creation operators matches the number of annihilators, hence $n = m$, and those where the momenta p and q are equal:

$$\langle 0_t | 0_t \rangle = \prod_p \frac{1}{|\alpha_p|^2} \sum_n \frac{1}{n!^2} \left(\frac{|\beta_p|^2}{|\alpha_p|^2} \right)^n \langle 0_{\rm in}| \, (b_{-p})^m (b_{-p}^\dagger)^n (a_p)^m (a_p^\dagger)^n |0_{\rm in}\rangle \tag{44}$$

Utilizing the relation (41), we finally obtain:

$$\langle 0_t | 0_t \rangle = \prod_p \frac{1}{|\alpha_p|^2} \sum_n \frac{1}{n!^2} \left(\frac{|\beta_p|^2}{|\alpha_p|^2} \right)^n \langle 0_{\rm in}| \, n! n! \, |0_{\rm in}\rangle \overset{(24)}{=} \prod_p \frac{1}{|\alpha_p|^2} \underbrace{\langle 0_{\rm in} | 0_{\rm in} \rangle}_{=1} \sum_{n=0}^{\infty} \left(\frac{|\alpha_p|^2 - 1}{|\alpha_p|^2} \right)^n \tag{45}$$

$$= \prod_p \frac{1}{|\alpha_p|^2} \sum_{n=0}^{\infty} \left(1 - \frac{1}{|\alpha_p|^2} \right)^n = \prod_p 1 = 1 \ , \tag{46}$$

where the geometric series has been evaluated according to $\sum_{n=0}^{\infty} x^n = 1/(1-x)$ for $|x| < 1$. This is fulfilled, since $|\alpha_p|^2 \geq 1$. Hence, formula (28) has been proven.
Next, at time t, we may calculate the number expectation value of, for instance, c-particles with momentum $\vec{p}$ in the initial vacuum state:

$$N_p(t) = \langle 0_{\rm in}| \, c_p^\dagger c_p \, |0_{\rm in}\rangle = \langle 0_{\rm in}| \left(\alpha_p^* a_p^\dagger + \beta_p^* b_{-p} \right) \left(\alpha_p a_p + \beta_p b_{-p}^\dagger \right) |0_{\rm in}\rangle \tag{47}$$

$$= |\beta_p(t)|^2 \langle 0_{\rm in}| \, b_{-p} b_{-p}^\dagger \, |0_{\rm in}\rangle = |\beta_p(t)|^2 \int_V \mathrm{d}^3 x \exp(i(\vec{p} - \vec{p})\vec{x}) = V |\beta_p(t)|^2 \ , \tag{48}$$

V being the volume of the space we are considering. Accordingly, the particle density is given by:

$$n_p(t) = |\beta_p(t)|^2 \; . \tag{49}$$

Hence, we see that a physical state, which coincides with the vacuum at time $t_{-\infty}$ may contain 'particles' at another time t and vice versa. This is the very gist of the Schwinger effect. In a sense, we create particles by redefining the concept of particle and vacuum in accordance with the external classical source. Note that the energy is not conserved, since we do not consider back reactions of quantum mechanical phenomena on the classical field. Or, technically speaking, the Hamiltonian explicitly depends on time through $A^3(t)$ and its dynamics does not arise within the quantum theory.
The probability of staying in the vacuum, i.e. not creating any particles, is calculated from (28):

$$|\langle 0_{\text{in}}|0_t\rangle|^2 = \prod_p \frac{1}{|\alpha_p(t)|^2} = \exp\left(-\sum_p \ln\left(1+|\beta_p(t)|^2\right)\right) = \exp\left(-\sum_p \ln(1+n_p(t))\right) \; , \tag{50}$$

where equations (24) and (49) have been employed. In the continuum limit, one replaces the sum by an appropriate momentum integral:

$$|\langle 0_{\text{in}}|0_t\rangle|^2 = \exp\left(-\frac{V}{(2\pi)^3}\int \mathrm{d}^3p \ln(1+n_p(t))\right) \tag{51}$$

If we were to take a very large volume V, the probability of remaining in the vacuum state would tend to zero. However, this will turn out to be quite cumbersome in experimental set-ups, since one needs to make sure that the electric field stays homogeneous throughout the volume.
Note that the whole problem boils down to finding the appropriate mode functions $f^{\pm}$ which determine the function E (20) that, in turn, yields the Bogolyubov coefficients β_p as stated in (25).

3 Constant Electric Field

In the following, in order to explicitly calculate this transition probability of the vacuum, we specify the external field by:

$$A^3(t) = -\frac{E}{k_0}\tanh(k_0 t) \to E^3(t) = -\partial_0 A^3(t) = \frac{E}{\cosh^2(k_0 t)} \; . \tag{52}$$

These two functions above can be seen in Figure 1. Equation (52) introduces a time scale $2/k_0$, the duration of the electric field pulse. In our case, scalar quantum electrodynamics, the mode functions can be given in terms of hypergeometric functions [GMM94], whose exact form is not presented, here. Instead we restrict ourselves to stating the results which emerge from these solutions in the following limits. We will be interested in late times $t \to \infty$, after the external field has been switched off again. For $k_0 \to 0$, the pulse length is going to infinity and we describe the physical situation of an electric field which is constant

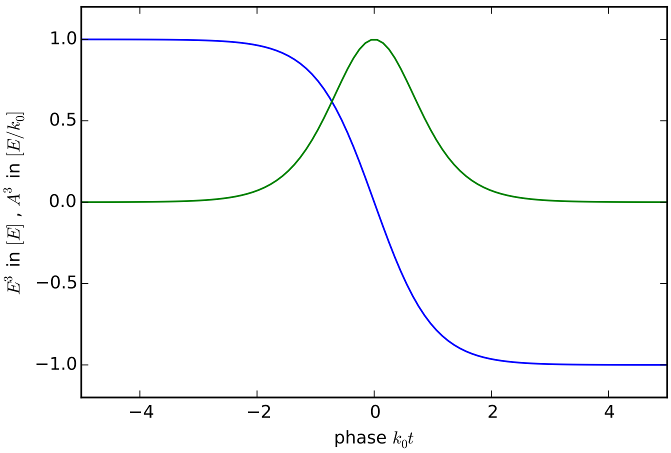

Figure 1: On equation (52): Field strength $E^3(t)$ given in units of its maximal value E (displayed in green). Vector potential $A^3(t)$ given in units of E/k_0 (displayed in blue).

in space and time. Later, we will take the result for the spectral particle density from [GMM94], calculated from the exact hypergeometric function.
But first, we may give some approximate solutions with the use of a WKB-approximation[1], following [Mar07]. Consider the following equation:

$$\ddot{f}_k^{\pm} + (\omega_k^2(t) - Q) f_k^{\pm} = 0 \ , \tag{53}$$

where we defined Q as:

$$Q = \frac{3}{4\omega_k^2}\,\dot{\omega}_k^2 - \frac{1}{2\omega}\,\ddot{\omega}_k \ . \tag{54}$$

Then the equation (53) is solved by the WKB mode functions:

$$f_k^{\pm}(t) = \frac{1}{\sqrt{2\omega_k(t)}} \exp\left(\pm i \int_{t_{-\infty}}^{t} \mathrm{d}t'\, \omega_k(t') \right) \ . \tag{55}$$

In fact, equation (53) is a good approximation to our problem, as long as Q is much smaller than ω_k^2:

$$\left| \frac{Q}{\omega_k^2} \right| = \left| \frac{5}{4} \frac{\tau^2}{(Y+\tau^2)^3} - \frac{1}{2(Y+\tau^2)^2} \right| \ , \tag{56}$$

with the dimensionless quantities $\tau := \sqrt{qE}t + k_3/\sqrt{qE}$ and $Y := (m^2 + k_\perp^2)/(qE)$. It is noted that the above fraction is small, whenever the modulus of τ is large. That means that the WKB-approximation is valid for any k value, if we go to sufficiently early or late times, while it breaks down in between. This behaviour is illustrated in Figure 2, where the fraction (56) has been plotted for three different Y-values: $Y = 1, 2, 3$. These numbers

[1] named after Wentzel, Kramers and Brillouin. [Mar07]

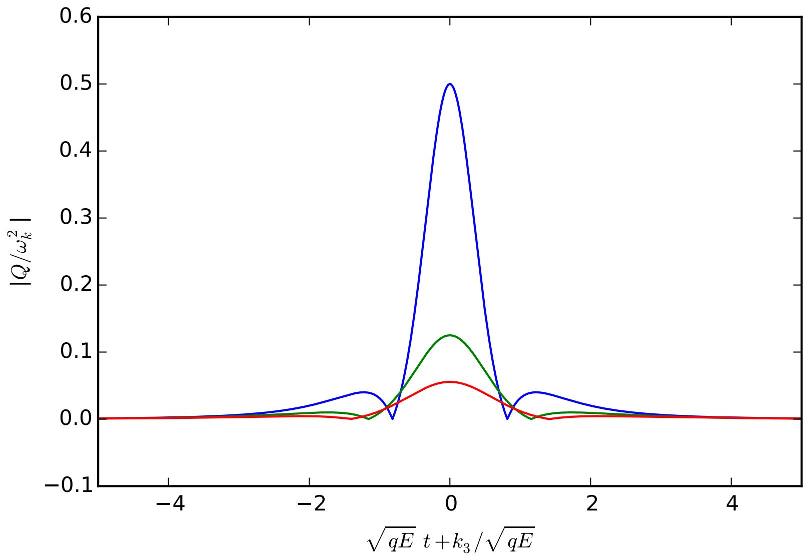

Figure 2: The quantity $\left|Q/\omega_k^2\right|$ is plotted against τ for three different values of Y: $Y = 1$ (blue), $Y = 2$ (green) and $Y = 3$ (orange). Similar to [Mar07].

have been chosen, since:

$$Y := \frac{k_\perp^2 + m^2}{qE} \geq \frac{m^2}{qE} \geq 1 \ , \tag{57}$$

assuming an electric field strength smaller than a critical field strength $E_{\text{crit}} = m^2/q$ which will be encountered again later on. We observe that the WKB-approximation becomes increasingly accurate for higher values of Y such that for large Y we recover the plain wave solution throughout all times again. This can be realised in the limit of vanishing external fields or for high transverse momenta. In both cases the relative change of ω with time becomes negligible and we are back to the case of standard scalar QFT.
In the WKB regime, the exponent is given by:

$$\int \mathrm{d}t'\, \omega_k(t') = \int_{t_{-\infty}}^{t} \mathrm{d}t'\, \sqrt{(qE)(Y + \tau^2)} = \int_{\tau_{-\infty}}^{\tau} \mathrm{d}\tau'\, \sqrt{Y + \tau'^2} \ , \tag{58}$$

using $\mathrm{d}\tau' = \mathrm{d}t'\, \sqrt{qE}$, since the coordinate is transformed linearly. This can be solved as follows [Mar07]:

$$\int \mathrm{d}t'\, \omega_k(t') = \frac{1}{2} \left[\tau' \sqrt{\tau' + Y} + Y \ln\left(\frac{\tau}{\sqrt{Y}} + \sqrt{1 + \frac{\tau^2}{Y}} \right) \right] \Bigg|_{\tau_{-\infty}}^{\tau} \tag{59}$$

However, these solutions (at very early and very late times respectively) have to be linearly combined such that they match the positive and negative frequency solutions at early times (22) again, i.e. they have to be consistent with the initial state. If the exact solution is not known one has to do further approximations about the situation at finite times in order

to match the different solutions in the opposite infinite time regimes.
Anyway, as said before, for now we will take the result from [GMM94], which is obtained from considering the exact hypergeometric solution in the $k_0 \to 0$ and $t \to \infty$ limit. That yields the spectral particle density [GMM94] after determining the mode functions, the function E (20) and the Bogolyubov coefficients (25):

$$n_p \approx \exp\left(\frac{-\pi(m^2 + p_\perp^2)}{|q|E}\right)\theta\left(|p^3 - |q|E/k_0|\right) \ , \tag{60}$$

where θ denotes the Heaviside step function. The particle density can then be written as follows, resulting in a mean pair production intensity:

$$n = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, n_p = T\frac{q^2E^2}{(2\pi)^3}\exp\left(-\frac{\pi m^2}{|q|E}\right) \longrightarrow I = \frac{q^2E^2}{(8\pi^3)}\exp\left(-\frac{\pi m^2}{|q|E}\right) \tag{61}$$

On the other hand, inserting the series expansion $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$ into (51), yields:

$$|\langle 0_{\text{in}}|0_t\rangle|^2 = \exp\left(-\frac{V}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int \mathrm{d}^3 p\ n_p^n\right) . \tag{62}$$

Utilising our result (61) from before, one finally obtains the *Schwinger formula*:

$$|\langle 0_{\text{in}}|0_t\rangle|^2 = \exp\left(-\frac{VT}{(2\pi)^3}(qE)^2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\exp\left(-\frac{n\pi m^2}{|q|E}\right)\right) . \tag{63}$$

Note that this is a non-perturbative result. It is not possible to Taylor expand our result around $|q|E = 0$, since the above function is not analytic at that very point. Hence, this result cannot easily be calculated from perturbation theory.
One can observe, that the particle production intensity at small electric field strengths is heavily suppressed due to the 'inner' exponential function. That means, at field strengths much smaller than some critical value E_{crit}, we may truncate the n-series at first order and Taylor expand the result as:

$$|\langle 0_{\text{in}}|0_t\rangle|^2 \approx 1 - V\cdot T\cdot\frac{q^2E^2}{(2\pi)^3}\exp\left(-\frac{\pi m^2}{|q|E}\right) = 1 - V\cdot T\cdot I \ . \tag{64}$$

Going back to SI-units, this critical field strength scale is given by [Mar07]:

$$E_{\text{crit}} = \frac{m^2c^3}{\hbar q} = \frac{mc^2}{q\lambda_C} \ . \tag{65}$$

In other words, in order to obtain non-negligible pair production intensities, the work which is performed by the classical field over the Compton wavelength λ_C must be of the order of the rest mass of the produced particle pair. [Mar07]
This seems to be a quite intuitive result. The Heisenberg uncertainty relation yields a certain length scale of quantum fluctuations, which is the Compton wavelength:

$$\Delta x \sim \frac{\hbar}{\Delta p} \sim \frac{\hbar}{mc} = \lambda_C \tag{66}$$

If, on this length scale, the two particles acquire energies on the order of their rest masses, they are likely to tunnel through the 'potential barrier' of particle creation.
However, once the field strength becomes comparable to that critical value, particle production is very strong and starting to influence the external field on a macroscopic scale. This is where Maxwell's theory of electrodynamics breaks down and non-linear corrections have to be taken into account. [HE36]
Note that, for the vacuum transition probability, the size of the spacetime volume $V \cdot T$ is crucial. If we were able to push this quantity to infinity, particle production would be guaranteed at any non-zero field strength and we cannot perform a Taylor expansion as we did above. Therefore, we choose a spacetime volume whose characteristic length scale is not too large, depending on the actual experimental set-up in use.

4 The Schwinger effect for Fermions

We now want to focus on the case of fermion quantum electrodynamics. The electron and the positron have much smaller masses than any charged scalar particle we know. Hence, the critical field strength is much smaller in QED, such that e^+e^--production will be the dominant effect. However, one needs to check if the above results apply to spinor particles as well. In this section the derivation of the Schwinger formula for fermions is briefly sketched.
Again, we start with the action of the system, given by the Dirac Lagrangian, which is invariant under a local $U(1)$-symmetry:

$$S = \int \mathrm{d}^4x \, \bar{\psi} \left(i\not{D} - m \right) \psi \ , \tag{67}$$

where $D_\mu = \partial_\mu + ieA_\mu$, and e being the electric charge. Again, the equation of motion is set by the variational principle. For ψ it reads:

$$\left(i\not{D} - m \right) \psi = 0 = \left(i\not{\partial} - e\gamma^3 A_3 - m \right) \psi \ . \tag{68}$$

It is advisable to define an auxiliary field χ in the following manner:

$$\psi = \left(i\not{\partial} - e\gamma^3 A_3 + m \right) \chi \ , \tag{69}$$

in order to transform equation (68) into a Klein-Gordon like second order equation for χ:

$$\begin{aligned} 0 &= \left(i\not{\partial} - e\gamma^3 A_3 - m \right) \left(i\not{\partial} - e\gamma^3 A_3 + m \right) \chi & (70) \\ &= \left(\partial^2 + m^2 + (eA_3)^2 + e\gamma^0\gamma^3 \dot{A}_3 - 2ieA_3\partial^3 \right) \chi \ , & (71) \end{aligned}$$

Solutions to the above equation can be obtained by choosing the following ansatz:

$$\chi^{\pm}_{\vec{p},r}(x) = e^{i\vec{p}\vec{x}} g^{\pm}(\vec{p}, t) R_r \ , \tag{72}$$

where $g^{\pm}$ denote our fermion mode functions and R_r the two positive eigenvalue eigenvectors of the matrix $\gamma^0\gamma^3$ which suffice to fix all degrees of freedom in our equation:

$$R_1 = (0, 1, 0, -1)^T \quad , \quad R_2 = (1, 0, 1, 0)^T \ . \tag{73}$$

Again, we will drop all vector arrows from our notation, keeping in mind that all following operators are depending on momentum direction. Plugging in the ansatz (72), we find the condition on the mode functions:

$$\ddot{g}^{\pm}(p,t) + \left(\omega^2(p,t) + ie\dot{A}_3(t)\right) g^{\pm}(p,t) = 0 \; . \tag{74}$$

For the external fields (52) the above equation can be solved in terms of hypergeometric functions again [GMM94]. In the limit of $k_0 \to 0$ and $t \to \infty$ one finds the same spectral particle density, as in the scalar case before [GMM94]:

$$n_p^f = n_p \approx \exp\left(\frac{-\pi(m^2 + p_\perp^2)}{|e|E}\right)\theta\left(\left|p^3 - |e|E/k_0\right|\right) \; . \tag{75}$$

However, note that this is the spectral density per spin projection. Hence, in order to obtain the total mean particle production intensity, one has to include a spin sum:

$$I^f = \frac{1}{T}\sum_{r=1}^{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\, n_p^f \approx 2\frac{e^2E^2}{(8\pi^3)}\exp\left(-\frac{\pi m^2}{|e|E}\right) \tag{76}$$

The rest of the analysis is actually very similar to what we have encountered before, when dealing with the scalar field, a few changes have to be made though. This is due to the fact that we are now dealing with anti-commutation relations, instead of commutators, and, as we have just seen above, because we obtained additional degrees of freedom, translating into two different spin projections for each particle.
Because of the anti-commutators, the Bogolyubov transformation has to be adapted, such that V_p^f now is an $SU(2)$ matrix:

$$\begin{pmatrix} a_p \\ b_{-p}^\dagger \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha_p^*(t) & -\beta_p(t) \\ +\beta_p^*(t) & \alpha_p(t) \end{pmatrix}}_{=:V_p^f} \begin{pmatrix} c_p \\ d_{-p}^\dagger \;, \end{pmatrix} \tag{77}$$

and hence its determinant reads:

$$1 = \det\left(V_p^f\right) = |\alpha_p|^2 + |\beta_p|^2 \; . \tag{78}$$

Also, the connection between two vacuum states changes. The instantaneous fermion vacuum is now related to the initial vacuum state via:

$$|0_t\rangle = \prod_{p,r} |\alpha_p(t)| \exp\left(-\frac{\beta_p}{\alpha_p}\, b_{-p}^\dagger a_p^\dagger\right) |0_{\mathrm{in}}\rangle \;\; , \tag{79}$$

where the normalisation factor changed due to the anti-commutation relations. The vacuum transition probability reads:

$$|\langle 0_{\mathrm{in}}|0_t\rangle|^2 = \prod_{p,r} |\alpha_p(t)|^2 = \exp\left(\sum_{p,r} \ln\left(1 - |\beta_p(t)|^2\right)\right) = \exp\left(\sum_{p,r} \ln(1 - n_p(t))\right) \tag{80}$$

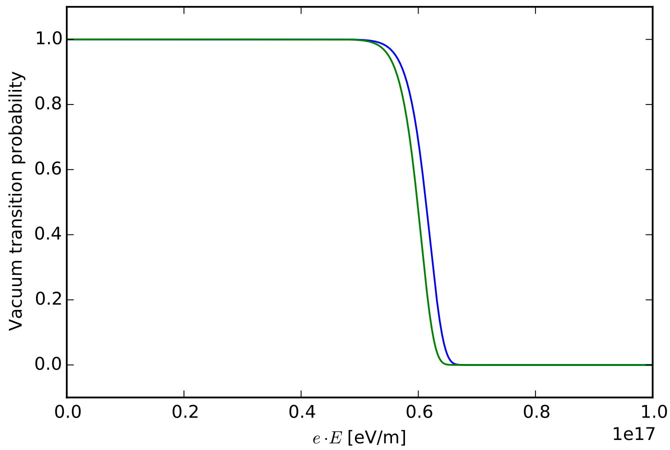

Figure 3: The probability for no particle creation is plotted against the constant electric charge times the electric field strength. We distinguish between electrons and positrons (green) and scalar particles with the electron mass and charge (blue).

Using $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, the fermion Schwinger formula reads:

$$|\langle 0_{\text{in}} | 0_t \rangle|^2 = \exp\left(-2 \frac{VT}{(2\pi)^3} (qE)^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{n\pi m^2}{|e|E} \right) \right) . \tag{81}$$

From equation (76) we see, that the critical field strength changes only through the different mass and charge compared to the scalar particles. For QED one obtains:

$$m_e c^2 \approx 511\,\text{keV}, \;\; e \approx 1.6 \times 10^{-19}\,\text{C} \Rightarrow E_{\text{crit}} \approx 1.3 \times 10^{18}\,\text{V/m} . \tag{82}$$

The vacuum transition probability, i.e. the probability for no particle creation at all, can be plotted against $e \cdot E$ for the scalar particles and the fermions respectively. As mentioned before, this result strongly depends on the spacetime volume. Let us assume a laser creates a pulse as described in equation (52). In order to get a very rough estimate, we use a set of typical laser values described in [AHR$^+$01]: the pulse duration is about 80 fs and its wavelength $\lambda \sim 0.1$ nm. It is focussed onto a spot of size λ^3 which basically describes the diffraction limit [AHR$^+$01]. The result is displayed in Figure 3. Note that due to the size of the spacetime volume particle creation starts around one order of magnitude below the critical field strength. Also, there is not much difference between fermions and scalar particles of equal mass and charge. In the next section, we will have a look at results from [AHR$^+$01] based on a more realistic calculation.

5 Experiments

Let us now contemplate a possible experimental set-up which might be able to measure the creation of electron-positron pairs in the vacuum. Earlier sections dealt with the description of pair production from an initial vacuum state in the presence of a homogeneous

	λ (nm)	E_0 (V/m)	$n_{\max}(t_>)$ (fm^{-3})	$N(t_>)$
Set Ia	0.15	1.3×10^{17}	4.6×10^{-13}	$\sim 10^3$
Set Ib	0.075	1.3×10^{17}	4.6×10^{-13}	$\sim 10^2$
Set IIa	0.15	1.3×10^{18}	7.2×10^{-10}	$\sim 10^6$
Set IIb	0.075	1.3×10^{18}	6.4×10^{-10}	$\sim 10^5$

Table 1: Two sets of lasers: Set I describes XFEL operating at 0.1 E_{crit}, Set II describes the strong field regime at E_{crit}. Both sets are divided into two different wavelengths (a and b), illustrating the experiment's dependence on the laser frequency. Columns three and four list the density and the amount of produced particles in a spot of size λ^3 at the time $t_>$ where they reach their maximum value. [AHR$^+$01]

electric field, which was assumed to be constant in time, i.e. stationary, in the limit of infinite pulse duration. However, it is very difficult to create field strengths on the order of the critical field strength calculated before, let alone stationary and homogeneous ones. Therefore, it might be rewarding to state the results of calculations, which are beyond the scope of this report, describing pair production by an oscillating field. This can be realised with a standing wave created by two counter-propagating high intensity laser beams. Still, the electric field will be assumed to be homogeneous on length scales much smaller than the laser's wavelength in the vicinity of the field maxima. Hence, the volume we are considering has to be chosen accordingly. In the following, we are considering numerical results for e^+e^--production in this situation obtained by Alkofer et al. in [AHR$^+$01]. They used two sets of field strengths, depicted in Figure 1. The first set describes pulsed X-ray free electron lasers (XFEL) operating on field strengths which are actually realisable. On the other hand, the second set illustrates the strong field regime on the order of the critical field strength. Also, Figure 1 yields the density and the amount of particles produced in a volume of λ^3 at the time $t_>$, when both reach their maximum. The pulse length is assumed to be around 80 fs. It is noted, that with present-day laser technology it should in principle be possible to create around one thousand particle pairs. In Figure 4, the expected number density of produced pairs is plotted against time. In the weak field case, particles are created and annihilated periodically with a frequency equal to the laser frequency. The maximum values during one cycle correspond to the numbers of 10^3 and 10^2 created pairs which have been stated in Figure 1. In the strong field regime, the same oscillation periods are encountered. However, the produced particles are not annihilated again after one laser cycle. Instead, their number density seems to saturate after a few periods.

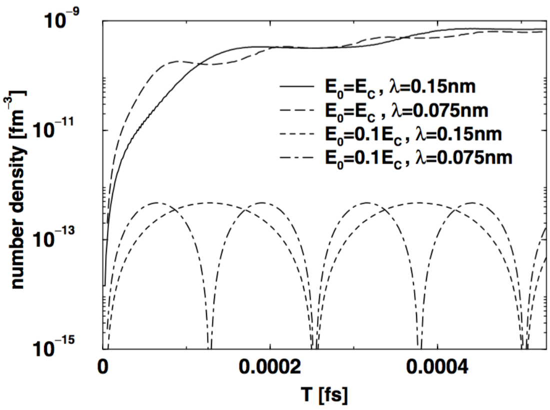

Figure 4: The number density of produced e^+e^--pairs is plotted against time for the different set-ups from Figure 1. Clearly one observes oscillations of the number densities following the laser cycle in the weak field regime, whilst the strong laser fields lead to a saturation of pair production. [AHR$^+$01]

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