# Quantum field theory in curved spacetime

## Quantisation of a scalar field with time-dependent mass

Arvid Weyrauch

Talk given on 11.11.2016

When starting to study quantum field theory in curved spacetimes considering a field with time dependent mass proves useful, since a variety of physically relevant scenarios -like a field in an expanding de Sitter spacetime- can be reparametrised such that the action becomes mathematically equivalent to a field in flat spacetime with time-dependent mass.

This is a report to the first talk in a seminar on quantum field theory in curved spacetime and gives an introduction to the quantisation of such a scalar field and some of the formal consequences. However, the physical interpretation of these results was covered by further talks and therefore is not reviewed here.

It is largely based on Chapter 6 of the lecture notes *Introduction to Quantum Fields in Classical Backrounds* by V. F. Mukhanov and S. Winitzki.

# 1 Classical scalar field

As the most basic example an action $S$ for a free scalar field $\chi$ is used. However, the mass is considered to be a function of the time-like coordinate $\eta$:

$$\begin{aligned} S &= \int \mathrm{d}^4x\, \mathcal{L} = \frac{1}{2}\int \mathrm{d}^4x\, \left(\partial_\alpha\chi\,\partial^\alpha\chi - m(\eta)\chi^2\right) \\ &= \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\mathrm{d}\eta\, \left((\dot\chi)^2 - (\nabla\chi)^2 - m(\eta)\chi^2\right) \quad , \end{aligned} \tag{1}$$

where the dot $\dot{}$ denotes the derivative with respect to $\eta$ and using the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$[1] for the coordinates $(\eta, \mathbf{x})$. The equation of motion is obtained with the Euler-Lagrange equation

$$\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\chi)}\right) - \frac{\partial\mathcal{L}}{\partial\chi} = 0 \quad , \tag{2}$$

implying

$$\ddot\chi - \Delta\chi + m(\eta)\chi = 0 \quad . \tag{3}$$

This can be treated more easily when expanding the field $\chi$ in Fourier modes $\chi_\mathbf{k}(\eta)$:

$$\chi(\mathbf{x},\eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}}\, \chi_\mathbf{k}(\eta)\, e^{i\mathbf{k}\mathbf{x}} \quad . \tag{4}$$

[1] The metric will not appear explicitly and therefore $\eta$ will always denote the time coordinate in this report.

Inserting 4 into the equation of motion 3 yields the equation of motion in Fourier space[2]

$$\ddot{\chi}_{\mathbf{k}} + \left(k^2 + m^2(\eta)\right)\chi_{\mathbf{k}} = 0 \quad , \tag{5}$$

which is the equation of motion for an oscillator with time-dependent frequency $\omega_{\mathbf{k}}(\eta)$

$$\omega_{\mathbf{k}}(\eta) = k^2 + m^2(\eta) \quad . \tag{6}$$

With any two linearly independent solutions $\{x_1(\eta), x_2(\eta)\}$of 5 the complex solution $v(\eta)$

$$v_{\mathbf{k}}(\eta) \equiv x_{\mathbf{k},1}(\eta) + i x_{\mathbf{k},2}(\eta) \tag{7}$$

can be defined and $\{v_{\mathbf{k}}(\eta), v^*_{\mathbf{k}}(\eta)\}$ is a basis, since the space of solutions is two dimensional. For simplicity the mode functions are chosen isotropic, meaning $v_{\mathbf{k}}(\eta) = v_k(\eta)$ only depends on $k = |\mathbf{k}|$. A useful concept is the Wronskian $W[x_1, x_2]$ of two functions $x_1(\eta)$ and $x_2(\eta)$

$$W[x_1, x_2] = \dot{x}_1 x_2 - x_1 \dot{x}_2 \quad . \tag{8}$$

If both arguments satisfy the same oscillator equation - like 5 - the Wronskian is time-independent

$$\partial_t W[x_1, x_2] = \ddot{x}_1 x_2 - x_1 \ddot{x}_2 = -\omega_{\mathbf{k}}(\eta) x_1 x_2 + x_1 \omega_{\mathbf{k}}(\eta) x_2 = 0 \quad . \tag{9}$$

Furthermore, the Wronskian is non-zero if and only if the arguments are linearly independent solutions. This is used in fixing the normalisation of the mode functions by requiring

$$\mathrm{Im}(\dot{v}_k v^*_k) = \frac{1}{2i}(\dot{v}_k v^*_k - \dot{v}^*_k v_k) = \frac{1}{2i} W[v_k, v^*_k] \overset{!}{=} 1 \quad . \tag{10}$$

Due to the properties of a Wronskian this can indeed be fixed for all times and any solution $v_k(\eta)$ satisfying this normalisation condition forms a basis of the complex solutions with its complex conjugate $v^*_k(\eta)$.

This also means complex coefficients $a^-_{\mathbf{k}}, a^+_{-\mathbf{k}}$ exist to express a general solution to 5 in terms of $v_k(\eta)$ and $v^*_k(\eta)$

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}}\left(a^-_{\mathbf{k}} v^*_k(\eta) + a^+_{-\mathbf{k}} v_k(\eta)\right) \quad . \tag{11}$$

Accordingly, solutions for 3 are given by

$$\begin{aligned} \chi(\mathbf{x}, \eta) &= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \ \left(a^-_{\mathbf{k}} v^*_k(\eta) + a^+_{-\mathbf{k}} v_k(\eta)\right) e^{i\mathbf{kx}} \\ &= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \ \left(a^-_{\mathbf{k}} v^*_k(\eta) e^{i\mathbf{kx}} + a^+_{\mathbf{k}} v_k(\eta) e^{-i\mathbf{kx}}\right) \quad . \end{aligned} \tag{12}$$

Since $\chi(\mathbf{x}, \eta)$ is real one finds that $(a^-_{\mathbf{k}})^* = a^+_{\mathbf{k}}$. Moreover,

$$a^-_{\mathbf{k}} = \sqrt{2}\frac{W[v_k, \chi_{\mathbf{k}}]}{W[v_k, v^*_k]} \quad , \tag{13}$$

showing that they are indeed time-independent. Furthermore, the coefficients depend on the choice of the mode function.

[2]This only works well if the mass is independent of $\mathbf{x}$. Otherwise $\chi$ transforms into different modes than $m(\eta, \mathbf{x})\chi$ usually making the quantisation very difficult.

## 2 Quantisation

A simple way to perform the quantisation is to promote $a^+_{\mathbf{k}}$ and $a^-_{\mathbf{k}}$ to the creation and annihilation operator $\hat{a}^+_{\mathbf{k}}$ and $\hat{a}^-_{\mathbf{k}}$. Additionally, their commutation relations need to be postulated:

$$[\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{\mathbf{k}'}] = \delta\left(\mathbf{k} - \mathbf{k}'\right) \qquad [\hat{a}^-_{\mathbf{k}}, \hat{a}^-_{\mathbf{k}'}] = [\hat{a}^+_{\mathbf{k}}, \hat{a}^+_{\mathbf{k}'}] = 0 \quad . \tag{14}$$

It schould be noted that, like before, the operators depend on the choice of the mode function.

Equation 12 now defines the field operator $\hat{\chi}(\mathbf{x}, \eta)$:

$$\hat{\chi}(\mathbf{x}, \eta) = \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{a}^-_{\mathbf{k}} v^*_k(\eta) e^{i\mathbf{k}\mathbf{x}} + \hat{a}^+_{\mathbf{k}} v_k(\eta) e^{-i\mathbf{k}\mathbf{x}}\right) \quad . \tag{15}$$

The momentum operator $\hat{\pi}(\mathbf{x}, \eta)$ is the time derivative of the field operator. Assuming the normalisation condition 10 the commutator $[\hat{\chi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)]$ can be calculated:

$$\begin{aligned} [\hat{\chi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)] = \frac{1}{2} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \, e^{i(\mathbf{k}\mathbf{x} + \mathbf{k}'\mathbf{y})} \\ \times \left[\hat{a}^-_{\mathbf{k}} v^*_k(\eta) + \hat{a}^+_{-\mathbf{k}} v_k(\eta), \hat{a}^-_{\mathbf{k}'} \dot{v}^*_{k'}(\eta) + \hat{a}^+_{-\mathbf{k}'} \dot{v}_{k'}(\eta)\right] \\ = \frac{1}{2} \int \frac{\mathrm{d}^3\mathbf{k}\mathrm{d}^3\mathbf{k}'}{(2\pi)^3} \, e^{i(\mathbf{k}\mathbf{x} + \mathbf{k}'\mathbf{y})} \left(v^*_k(\eta) \dot{v}_{k'}(\eta) - v_k(\eta) \dot{v}^*_{k'}(\eta)\right) [\hat{a}^-_{\mathbf{k}}, \hat{a}^+_{-\mathbf{k}'}] \, . \end{aligned} \tag{16}$$

By using the commutation relations 14 one integration can be performed

$$\begin{aligned} [\hat{\chi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)] &= i \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \, e^{i\mathbf{k}(\mathbf{x} - \mathbf{y})} \frac{\dot{v}_k(\eta) v^*_k(\eta) - \dot{v}^*_k(\eta) v_k(\eta)}{2i} \\ &= i \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \, e^{i\mathbf{k}(\mathbf{x} - \mathbf{y})} \mathrm{Im}(\dot{v}_k(\eta) v^*_k(\eta)) \quad . \end{aligned} \tag{17}$$

According to the normalisation condition 10, $\mathrm{Im}(\dot{v}_k(\eta) v^*_k(\eta)) = 1$. The remaining intergral yields the usual commutation relation

$$[\hat{\chi}(\mathbf{x}, \eta), \hat{\pi}(\mathbf{y}, \eta)] = i\delta(\mathbf{x} - \mathbf{y}) \quad . \tag{18}$$

The Hamiltonian of the system is

$$\hat{H}(\eta) = \frac{1}{2} \int \mathrm{d}\mathbf{x} \left(\hat{\pi}^2 + (\nabla\hat{\chi})^2 + m^2(\eta)\hat{\chi}^2\right) \quad . \tag{19}$$

Since the Hamiltonian is explicitly time-dependent there is no energy conservation.

The vacuum state $|0\rangle$ is defined the eigenstate of the annihilation operator with eigenvalue 0:

$$a^-_{\mathbf{k}} |0\rangle = 0 \quad , \tag{20}$$

while particle states $|m_{\mathbf{k}_1}, n_{\mathbf{k}_2}, ...\rangle$ are created by an appropriate stack of creation operators acting on the vacuum state:

$$|m_{k_1}, n_{k_2}, ...\rangle = \frac{1}{\sqrt{m! n! \ldots}} \left( (\hat{a}_{\mathbf{k}_1})^m \, (\hat{a}_{\mathbf{k}_2})^n \ldots \right) |0\rangle \quad . \tag{21}$$

It is important to note that the creation an annihilation operators depend on the choice of the mode function. The physical interpretation of this fact is not part of this report, however a few of the mathematical consequences will be discussed next.

# 3 Bogolubov transformation

Let $u_k(\eta)$ and $v_k(\eta)$ be two mode functions. Since $\{u_k(\eta), u_k^*(\eta)\}$ is a basis of the space of solutions $v_k^*(\eta)$ can be expressed in terms of $u_k(\eta)$ and $u_k^*(\eta)$

$$v_k^*(\eta) = \alpha_k u_k^*(\eta) + \beta_k u_k(\eta) \quad . \tag{22}$$

This is called a Bogolubov transformation and $\alpha_k$ and $\beta_k$ are called the Bogolubov coefficients. A look at the normalisation of $v_k^*(\eta)$ given by equation 10 reveals a constraint on $\alpha_k$ and $\beta_k$

$$\begin{aligned} \mathrm{Im}\left(\dot{v}_k v_k^*\right) &= \mathrm{Im}\left(|\alpha_k|^2 \dot{u}_k(\eta) u_k^*(\eta) + |\beta|^2 \dot{u}_k^*(\eta) u_k(\eta)\right) \\ &= \left(|\alpha|^2 - |\beta|^2\right) \mathrm{Im}\left(\dot{u}_k(\eta) u_k^*(\eta)\right) \quad . \end{aligned} \tag{23}$$

Since both $u_k(\eta)$ and $v_k(\eta)$ are assumed to satisfy the normalisation condition 10 this yields

$$|\alpha|^2 - |\beta|^2 = 1 \quad . \tag{24}$$

Furthermore, the mode expansion of the field operator can be done with both mode functions $u_k(\eta)$ and $v_k(\eta)$ and their respective creation and annihilation operators $\{\hat{b}_{\mathbf{k}}^-, \hat{b}_{\mathbf{k}}^+\}$ and $\{\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}}^+\}$

$$\begin{aligned} \hat{\chi}(\mathbf{x}, \eta) &= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{a}_{\mathbf{k}}^- v_k^*(\eta) e^{i\mathbf{kx}} + \hat{a}_{\mathbf{k}}^+ v_k(\eta) e^{-i\mathbf{kx}}\right) \\ &= \frac{1}{\sqrt{2}} \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left(\hat{b}_{\mathbf{k}}^- u_k^*(\eta) e^{i\mathbf{kx}} + \hat{b}_{\mathbf{k}}^+ u_k(\eta) e^{-i\mathbf{kx}}\right) \quad . \end{aligned} \tag{25}$$

These have to be equal for all momenta, therefore

$$\hat{a}_{\mathbf{k}}^- v_k^*(\eta) + \hat{a}_{-\mathbf{k}}^+ v_k(\eta) = \hat{b}_{\mathbf{k}}^- u_k^*(\eta) + \hat{b}_{-\mathbf{k}}^+ u_k(\eta) \quad . \tag{26}$$

By rewriting $v_k(\eta)$ in terms of $u_k(\eta)$, $\hat{b}_{\mathbf{k}}^-$ and $\hat{b}_{\mathbf{k}}^+$ can be expressed in terms of $\hat{a}_{\mathbf{k}}^-$ and $\hat{a}_{\mathbf{k}}^+$:

$$\hat{b}_{\mathbf{k}}^- = \alpha_k \hat{a}_{\mathbf{k}}^- + \beta_k^* \hat{a}_{-\mathbf{k}}^+ \qquad \hat{b}_{\mathbf{k}}^+ = \alpha_k^* \hat{a}_{\mathbf{k}}^+ + \beta_k \hat{a}_{-\mathbf{k}}^- \quad . \tag{27}$$

If $u_k(\eta)$ and $v_k(\eta)$ are not identical their creation and annihilation operators are not identical either. Since the annihilation operator determines the vacuum state this also means there are two non identical vacuum states $|0\rangle_a$ and $|0\rangle_b$

$$\hat{b}_{\mathbf{k}}^- |0\rangle_b = 0 \qquad \hat{a}_{\mathbf{k}}^- |0\rangle_a = 0 \quad . \tag{28}$$

This results in the question whether the vacuum state $|0\rangle_a$ is a particle state in terms of $u_k(\eta)$.

The particle number operator for b-particles is given by $\hat{N}_k^{(b)} = \hat{b}_{\mathbf{k}}^{+}\hat{b}_{\mathbf{k}}^{-}$. The expectation value in the a-vacuum state $|0\rangle_a$ can be calculated by applying equation 27

$$\begin{aligned} \langle 0|_a \hat{N}_k^{(b)} |0\rangle_a &= \langle 0|_a \hat{b}_{\mathbf{k}}^{+}\hat{b}_{\mathbf{k}}^{-} |0\rangle_a \\ &= \langle 0|_a (\alpha_k^* \hat{a}_{\mathbf{k}}^{+} + \beta_k \hat{a}_{-\mathbf{k}}^{-})(\alpha_k \hat{a}_{\mathbf{k}}^{-} + \beta_k^* \hat{a}_{-\mathbf{k}}^{+})|0\rangle_a \quad . \end{aligned} \tag{29}$$

The terms containing on $\alpha_k$ drop, since they annihilate with the vacuum to the left or right, leaving only a term solvable with the commutation relations 14

$$\begin{aligned} \langle 0|_a \hat{N}_k^{(b)} |0\rangle_a &= |\beta_k|^2 \langle 0|_a \hat{a}_{-\mathbf{k}}^{-} \hat{a}_{-\mathbf{k}}^{+} |0\rangle_a \\ &= |\beta_k|^2 \left[\hat{a}_{-\mathbf{k}}^{-}, \hat{a}_{-\mathbf{k}}^{+}\right] = |\beta_k|^2 \delta^{(3)}(0) \quad . \end{aligned} \tag{30}$$

The $\delta^{(3)}(0)$ represents the volume and would be finite if considering a finite volume. Therefore, $|\beta_k|^2$ can be considered the non-zero density of b-particles with momentum $k$ in the a-vacuum state:

$$n_k = |\beta_k|^2 \tag{31}$$

# 4 Summary

The quantisation of a free scalar field with time-dependent mass is formally very similar to a quantisation with time-independent mass. However, the Hamiltonian is explicitly time-dependent, thus breaking energy conservation.

The quantisation can be performed such that it allows the choice of different mode functions resulting in different operators, vacuum states and definitions of particles.

A vacuum state resulting from a certain mode function will usually contain particles when considered with another choice of mode functions. The different mode functions are connected by the Bogolubov transformation and the Bogolubov coefficients.

It will turn out that the changes of energy can be considered to be a result of gravitational interaction and that the most suitable definition of the vacuum state is time-dependent.