# Quantum Field Theory in the de Sitter spacetime

**REPORT**

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# 1 Introduction

The de Sitter spacetime is a toy model for cosmological inflation in the sense that it describes inflation partially, namely for some period of time. It is because the inflation after some time ends. But the de-Sitter spacetime evolves eternally. There should be some mechanism which stops the inflation. In the de Sitter spacetime model this is not taken into account. In this report we start with geometrical definition and some properties of the de-Sitter space. Then we build scalar quantum field theory in this space. The main difference from the usual scalar field theory is that we have here an effective mass which is time dependent. This dependence is due to the interaction with gravitational field. Then we introduce preferred vacuum state in the de-Sitter space-time which is called the Bunch-Davies vacuum. Afterwards we have a look at quantum fluctuations in the de-Sitter space-time and calculate power spectrum.

# 2 de-Sitter space

De-Sitter space is an geometrical object in a Minkowski space. In order to have some intuition for curved spaces we start with the simple example, i.e with 2-sphere, $\mathbb{S}^2$ embedded in Euclidean space- $\mathbb{R}^3$. $\mathbb{S}^2$ is given by the following equation

$$X^2 + Y^2 + Z^2 = R^2 \tag{1}$$

where X,Y,Z are the coordinates for the background $\mathbb{R}^3$ space. Its metric is as following

$$ds^2 = \delta_{ij} dX^i dX^j \tag{2}$$

here $\delta_{ij}$ is an identity matrix and i,j=X,Y,Z

Now we want to derive the expression of the metric for the $\mathbb{S}^2$. In order to do this we perform coordinate transformations to a spherical coordinates on the $\mathbb{S}^2$. We want to get rid of the redundant degree of freedom.

$$X = Rsin\theta cos\phi;\ Y = Rsin\theta sin\phi;\ Z = Rcos\theta \tag{3}$$

after differentiating these formulas and plugging in the expression for the background metric we get the desired metric for the $\mathbb{S}^2$:

$$ds^2 = R^2 d\theta + R^2 sin^2\theta d\phi^2 \tag{4}$$

Using this technique we can derive metrics for higher dimensional objects-$3-sphere$, $4-sphere$ etc. These objects are homogeneous, isotropic and has positive constant curvature. This geometry is a part of Riemannian geometry. In this geometry the signs of the components of the metric are the same. The invariance group of a sphere is a rotational group of ambient space. For example, in $\mathbb{S}^2$ case it is SO(3) group.

Now we consider the Lorentzian geometry where metric components has different signs. Namely, we have time component. In this case as a background space we have Minkowski space, $\mathbb{M}^3$ and a hyperbola embedded in this space. The metric of the three dimensional Minkowski space is as following:

$$ds^2 = -dT^2 + dX^2 + dY^2 \tag{5}$$

and the equation of hyperbola

$$-T^2 + X^2 + Y^2 = L^2 \tag{6}$$

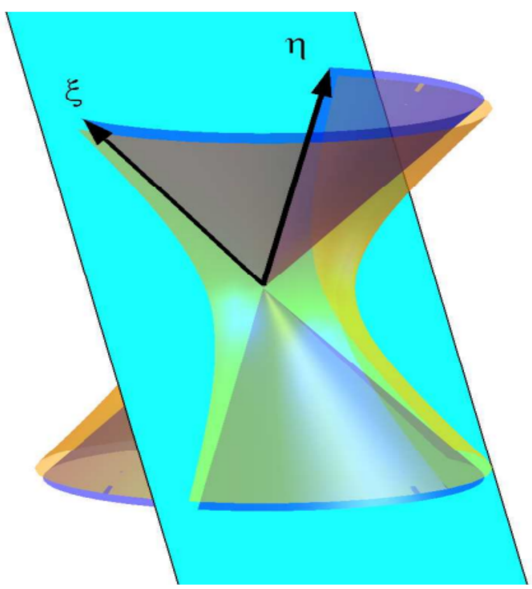
Now we induce the metric for the hyperbola embedded in $\mathbb{M}^3$ using the metric for $\mathbb{M}^3$. Again we perform coordinate transformations.

$$T = Lsinh(t/L);\ X = Lcosh(t/L)cos\phi;\ Y = Lcosh(t/L)sin\phi \tag{7}$$

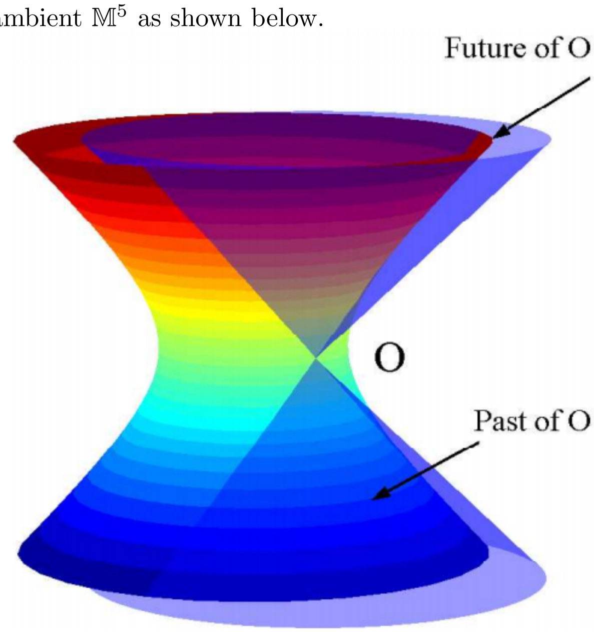
plugging these new coordinates in the $\mathbb{M}^3$ metric we get an expression for the metric of hyperbola

$$ds^2 = -dt^2 + L^2cosh(t/L)d\phi^2 \tag{8}$$

here the $L^2cosh(t/L)$ factor in the second term looks like a radius which oscillates depending on the values of t. So, the hyperbola shrinks and expands. One can generalize this to higher dimensional hyperbolas. The $4-hyperbola$, $\mathbb{H}^4$ embedded in 5 dimensional Minkowski space, $\mathbb{M}^5$ is called de-Sitter space(or space-time), $de\mathbb{S}_4$. As in the spheres case the $de\mathbb{S}_4$ possesses the symmetry group of the ambient space, namely, Lorentz group $SO(1,4)$ of $\mathbb{M}^5$.




The $de$ $\mathbb{S}_4$is described above in the figure. The geodesic on de-Sitter space is the intersection of a 2-plane, which passes through the center, with the space. The casual structure of the de-Sitter space is determined by the lightcone of the ambient $\mathbb{M}^5$ as shown below.




Now let's consider Einstein's field equations for this metric. In general de-Sitter spacetime the Ricci tensor and Ricci scalar are calculated as following:

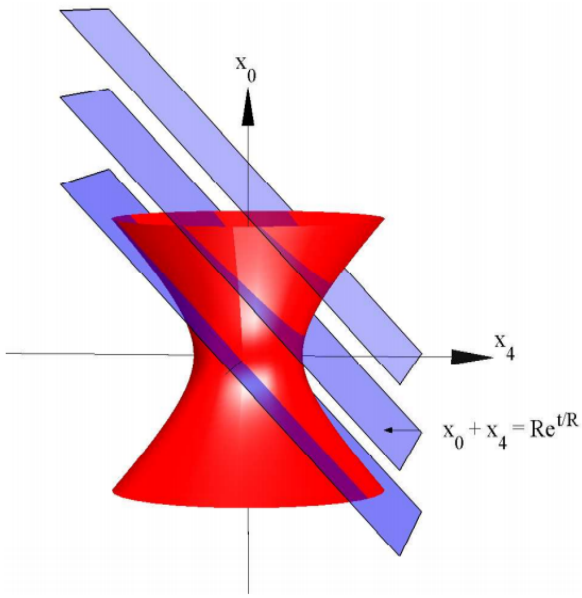
$$R_{ab} = \frac{d-1}{L^2} g_{ab};\ R = \frac{d(d-1)}{L^2} \tag{9}$$

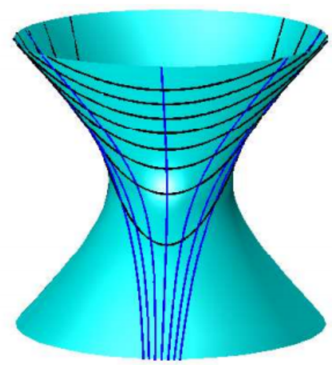
In $d = 4$ case we get $R = \frac{12}{L^2}$. If we plug these in the Einstein's equations for the vacuum with cosmological constant

$$R_{ab} - \frac{1}{2}R + \Lambda g_{ab} = 0 \tag{10}$$

We get $\Lambda = \frac{3}{L^2}$. Hence we say that the de-Sitter space is a solution to Einstein's field equations in vacuum with positive cosmological constant.

Depending on chosen coordinate system the metric for the de-Sitter space can have different expressions. For our purpose we choose such coordinate system which spatial part evolves exponentially. Our co-ordinate system is constructed by intersection of inclined planes with with de-Sitter space which satisfies $x_0 + x_4 = Re^{t/R}$(as shown in the figures below)





In this co-ordinate system the metric is expressed as follows:

$$ds^2 = dt^2 - exp(\frac{2t}{R})d\vec{x} \tag{11}$$

This metric is the spatially flat FRW metric. The FRW spaces is the most general homogeneous and isotropic solutions to the Einstein's equation. The cosmological constant is distributed homogeneously and isotropically. Now we rewrite the metric above in terms of cosmological parameters, namely the proper FRW metric.

$$t \equiv \frac{t}{2}, R \equiv \frac{1}{H}$$

$$ds^2 = dt^2 - a^2(t)d\vec{x} \tag{12}$$

here $a(t) = a_o exp(Ht)$, $H = \frac{\dot{a}}{a} > 0$ The de-Sitter metric can be derived using Eintein's equation for a universe filled with the homogeneous matter(perfect fluid) with the equation of state $p = -\varepsilon$. The presence of this kind of matter is equivalent to cosmological constant. This comes from the conservation of energy:

$$\frac{d\varepsilon}{dt} = -3(\varepsilon + p)\frac{\dot{a}}{a} = 0 \tag{13}$$

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu} = \varepsilon g^{\mu\nu}$$

The 0-0th component of Einstein's equation for a flat FRW spacetime yields the following equation:

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\varepsilon}{3} \tag{14}$$

which has a solution: 0

$$a(t) = a_o exp(t\sqrt{\frac{8\pi G\varepsilon}{3}}) \equiv a_o exp(Ht)$$

where $H = \sqrt{\frac{8\pi G\varepsilon}{3}}$ is a time independent Hubble parameter.

*Incompleteness of the co-ordinates (t,x).* The coordinates (t,x) vary in the $(-\infty, +\infty)$ interval and yet do not cover the entire de-Sitter space-time. In order to show the incompleteness explicitly let's consider a freely falling observer. Its trajectory x(t) is timelike. Now let's calculate its proper time.

$$\tau[x(t)] = \int dt\sqrt{1 - a(t)^2\dot{x}^2} \tag{15}$$

x(t) extremizes the proper time:

$$\frac{\delta\tau}{\delta x(t)} = 0 \Rightarrow \tag{16}$$

$$\frac{d}{dt}\frac{a(t)^2\dot{\vec{x}}}{\sqrt{1 - a(t)^2\dot{\vec{x}}^2}} \equiv \frac{d}{dt}\vec{p} = 0$$

$$a^2\dot{x}^2 = \frac{p^2}{p^2 + a^2}$$

If $\dot{x} \neq 0 \Rightarrow p \neq 0$ then

$$\tau_o = \int_{-\infty}^{0} dt\sqrt{1 - a(t)^2\dot{x}^2} = \int_{-\infty}^{0} \frac{a(t)dt}{\sqrt{p^2 + a^2}} = H^{-1}sinh^{-1}\frac{1}{p} < \infty$$

We see that in an infinite range of t and x proper time $\tau$ is finite. But this incompleteness of the co-ordinates is irrelevant in our case. Because the entire de-Sitter space is not compatible with the inflation as was mentioned previously. We use only a small portion of the de-Sitter space for the inflation.

*Horizons.* Every observer has its own horizon depending on its position in the de-Sitter space. The lightray satisfies $a(t)^2\dot{x}^2 = 1$. Its solution is

$$|\vec{x}| = \frac{1}{H}(e^{-Ht_o} - e^{-Ht})$$

We see that all lightrays emitted at $t_o$ asymptotically approach the sphere

$$|\vec{x}| = r_{max}(t_o) \equiv H^{-1}e^{-Ht_o}$$

- horizon scale. The distances beyond this sphere isn't accesible to the observer.

## 3 Scalar Field

The FRW space is conformally flat space. It means that by a conformal transformation one can express its metric as $g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu}$, $\eta_{\mu\nu}$ is the Minkowski metric and $\Omega(x) \neq 0$ is some function. Then let's write the FRW metric in a explicitly conformal flat form. In order to do this one conformally transforms the time co-ordinate.

$$\eta(t) = \int_0^t dt' e^{-Ht'} = -\frac{e^{-Ht}}{H}$$

then $a(\eta) = -\frac{1}{H\eta}$ and the metric becomes

$$ds^2 = a(\eta)[d\eta^2 - d\vec{x}^2]$$

the conformal time ranges in $(-\infty, 0)$ interval when the $t \in (-\infty, +\infty)$.

The so called Minimal coupling to gravity action is as follows:

$$S = \int \sqrt{-g} d^4x [\frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi)] \tag{17}$$

here $V(\phi) = \frac{1}{2}m^2\phi^2$. Then we derive the equation of motion(e.o.m) from this action:

$$g^{\mu\nu}\phi_{,\mu\nu} + \frac{1}{\sqrt{-g}}(g^{\mu\nu}\sqrt{-g})_{,\mu}\phi_{,\nu} + \frac{\partial V(\phi)}{\partial \phi} = 0$$

if we plug in $g_{\mu\nu} = a^2\eta_{\mu\nu}$, $g^{\mu\nu} = a^{-2}\eta^{\mu\nu}$, $\sqrt{-g} = a^4$ we get :

$$\phi_{,\mu}^{;\mu} + 2\frac{\dot{a}}{a}\phi' + m^2a^2\phi = 0 \tag{18}$$

At this point it is convenient to introduce an auxiliary field $\zeta \equiv a(\eta)\phi$. Then eq.(18) becomes :

$$\zeta'' - \Delta\zeta + (m^2a^2 - \frac{a''}{a})\zeta = 0 \tag{19}$$

where $\zeta'' \equiv \frac{\partial^2\zeta}{\partial\eta^2}$. If we compare this equation with the Klein-Gordon equation in Minkowski space we see that these two are the same except the mass term. We call this mass term as an effective mass $m_{eff}^2(\eta) = m^2a^2 - \frac{a''}{a}$. The procedure of replacing the field by the auxiliary field and thus having the effective mass leads to simplification. Namely, the dynamics of $\phi$ scalar field in the present of gravity is equivalent to the dynamics of v field in the Minkowski space with time-dependent effective mass which contains all information about the effects of gravity on $\phi$ field.

From equation (19) we see that because of $\Delta\zeta$ term the fields or oscillators are coupled to each other. In order to decouple them we Fourier expand the v field.

$$\zeta(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \zeta_k(\eta) e^{i\vec{k}\vec{x}} \tag{20}$$

if we plug this in the (19) we get equation for the decoupled modes:

$$\zeta_k'' + \omega_k^2(\eta)\zeta_k = 0 \tag{21}$$

where $\omega_k^2 = k^2 + m^2 a^2(\eta) - \frac{a''}{a}$.The general solution to this equation can be expressed in terms of mode functions:

$$\zeta(\eta) = \frac{1}{\sqrt{2}}[a_{\vec{k}}^- v_k^*(\eta) + a_{-\vec{k}}^+ v_k(\eta)] \tag{22}$$

where the mode functions $v$ and $v^*$ are linearly independent and form a basis in the space of complex solutions. The $a_{\vec{k}}^{\pm}$ are complex constants of integration. From the realness of $\zeta$ and from equation(22) it follows that $a_{\vec{k}}^+ = (a_{\vec{k}}^-)^*$.

If we insert the eq.(22) in the eq.(20) we express the $\zeta$ field in terms of the mode functions and the integration constants which in the end turns out to be creation and annihilation operators.

$$\zeta(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}(e^{i\vec{k}\vec{x}} v_k^*(\eta) a_{\vec{k}}^- + e^{-i\vec{k}\vec{x}} v_k(\eta) a_{\vec{k}}^+) \tag{23}$$

The choice of the mode functions is arbitrary and the $a_{\vec{k}}^{\pm}$ can expressed in terms of them.

$$a_{\vec{k}}^- = \sqrt{2}\frac{v_k'\zeta_{\vec{k}} - v_k\zeta_{\vec{k}}'}{v_k' v_k^* - v_k v_k^{*\prime}} \tag{24}$$

*Quantization of the scalar field.* One can use the standard canonical procedure to quantize the $\zeta$ field by introducing equal time commutation relations:

$$[\hat{\zeta}(\vec{x},\eta), \hat{\pi}(\hat{y},\eta)] = i\delta(\vec{x} - \vec{y}) \tag{25}$$

where $\hat{\pi} = \frac{d\hat{\zeta}}{d\eta} \equiv \hat{\zeta}'$ is a canonical momentum. The Hamiltonian is given by:

$$\hat{H}(\eta) = \frac{1}{2}\int d^3(\hat{\pi}^2 + (\vec{\nabla}\hat{\zeta})^2 + m_{eff}^2(\eta)\hat{\zeta}^2) \tag{26}$$

Another way of doing it is to use mode expansion of the quantum field(field operator) like a classical one. In this case one uses the mode expansion of the classical field and promotes the integration constants $a_{\vec{k}}^{\pm}$ to the $\hat{a}_{\vec{k}}^{\pm}$

$$\hat{\zeta}(\vec{x},\eta) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2}}(e^{i\vec{k}\vec{x}} v_k^*(\eta) \hat{a}_{\vec{k}}^- + e^{-i\vec{k}\vec{x}} v_k(\eta) \hat{a}_{\vec{k}}^+) \tag{27}$$

In this case the equation of motion for the $\zeta$ field is satisfied for the mode functions.

$$v_k'' + \omega^2(\eta)v_k = 0 \tag{28}$$

here, $\omega_k(\eta) \equiv \sqrt{k^2 + m_{eff}^2}$, $m_{eff}^2(\eta) = m^2 a^2 - \frac{a''}{a}$. The operators $\hat{a}_{\vec{k}}^{\pm}$ satisfy the usual creation and annihilation commutation relations:

$$[\hat{a}_{\vec{k}}^-, \hat{a}_{\vec{k}}^+] = \delta(\vec{k} - \vec{k}') \tag{29}$$

other commutations vanish.
Now let's focus on the eq.(28). If we plug the $a = e^{Ht}$ expression for the scale factor in the eq.(28):

$$v_k'' + [k^2 - (2 - \frac{m^2}{H^2})\frac{1}{\eta^2}]v_k = 0 \tag{30}$$

here we see that, $\omega_k^2(\eta) \equiv k^2 + m_{eff}^2 = k^2 + (\frac{m^2}{H^2} - 2)\frac{1}{\eta^2}$ can be imaginary. It can be imaginary- $\omega_k^2(\eta) < 0$ if $m^2 < 2H^2$. In the ealier times of the inflation which is described by a patch of the de-Sitter space $H \gg m$. m is the mass of elementary particles.
The solution to eq.(30) can be found by using Bessel equation. Namely, the equation can be brought into the Bessel equation by changing the variable and then solve the Bessel equation which is as follows:

$$x^2\frac{d^2y(x)}{dx^2} + x\frac{dy(x)}{dx} + (x^2 - n^2) = 0 \tag{31}$$

and the general solution to this equation is

$$y(x) = AJ_n(x) + BY_n(x) \tag{32}$$

where $J_n(x), Y_n(x)$ are the first kind and second kind Bessel functions respectively. The first kind Bessel function can be defined by its series expansion around $x = 0$

$$J_\alpha(x) = \sum_{m=0}^{+\infty}\frac{(-1)^m}{m!\Gamma(m+\alpha+1)}(\frac{x}{2})^{2m+\alpha} \tag{33}$$

it is finite at the origin($x = 0$) for integer or positive $\alpha$ and diverge as x approaches zero for negative non-integer $\alpha$.

The Bessel function of the second kind can be defined in terms of the first one,

$$Y_\alpha(x) = \frac{J_\alpha(x)cos(\alpha\pi) - J_{-\alpha}(x)}{sin(\alpha\pi)} \tag{34}$$

-for non-integer $\alpha$. This function(group of functions) has a singularity at the origin(x=0) and are multivalued. Now using this facts we will solve the eq.(30). First we need to bring it into the Bessel equation.

We change the variable as $\eta \to x \equiv k|\eta|$ and the function $v_k \equiv \sqrt{x}y(x) = \sqrt{k|\eta|}y(k|\eta|)$. If we substitute these changes in the eq.(30) we get the Bessel equation for the function y(x). The explicit derivation is as follows: first let's calculate the second order derivative in the eq. (30),

$$\frac{dv_k}{d\eta} = \frac{d(\sqrt{k|\eta|y(k|\eta|)})}{d\eta} = \frac{\eta}{|\eta|}\frac{k}{2\sqrt{k|\eta|}}y(k|\eta|) + k\frac{\eta}{|\eta|}\sqrt{k|\eta|}\frac{dy(k|\eta|)}{d(k|\eta|)}$$

$$\frac{d^2v_k}{d\eta^2} = -\frac{\eta^2}{(|\eta|)^2}\frac{k^2}{4(k|\eta|)^{3/2}}y(k|\eta|) + \frac{\eta}{|\eta|}\frac{k}{2\sqrt{k|\eta|}}\frac{dy(k|\eta|)}{d(k|\eta|)}\frac{d(k|\eta|)}{d\eta} + k^2\frac{\eta^2}{(|\eta|)^2}\frac{1}{2\sqrt{k|\eta|}}\frac{dy(k|\eta|)}{d(k|\eta|)}$$

$$+k\frac{\eta}{|\eta|}\sqrt{k|\eta|}\frac{d^2y(k|\eta|)}{d(k|\eta|)^2}\frac{d(k|\eta|)}{d\eta}=k^2\sqrt{x}\frac{d^2y}{dx^2}+k^2\frac{1}{\sqrt{x}}\frac{dy(x)}{dx}-k^2\frac{1}{4x^{3/2}}y(x)$$

if we plug the expression for the second order derivative in the eq.(30) and take into account the change of the function and the variable we get:

$$k^2\sqrt{x}\frac{d^2y(x)}{dx^2}+\frac{k^2}{\sqrt{x}}\frac{dy(x)}{dx}-\frac{k^2}{4x^{3/2}}y(x)+[k^2-(2-\frac{m^2}{H^2})\frac{k^2}{x^2}]\sqrt{x}y(x)=0$$

Finally multiplying by $\frac{x^2}{\sqrt{x}}$ we obtain the desired Bessel eqaution for the $y(x)$:

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}+[x^2-n^2]y(x)=0 \tag{35}$$

where $n=\sqrt{\frac{9}{4}-\frac{m^2}{H^2}}$ Now using the solution of the Bessel equation which was given above and taking into account that $y(x)=\frac{v_k(\eta)}{\sqrt{x=k|\eta|}}$ we can write the general solution for the $v_k$.

$$v_k(\eta)=\sqrt{k|\eta|}[AJ_n(k|\eta|)+BY_n(k|\eta|)] \tag{36}$$

The normalization condition for the mode function $Im=(v^*v_k')=1$ constrains the the constants A, B:

$$AB^*-A^*B=\frac{i\pi}{k}$$

Now we want to look at asymptotics of the mode function $v_k$. The mathematically rigorous way to do this is looking at the asymptotics of the Bessel functions. But one can consider this by looking at the asymptotics of the eq. (30) itself and solving it.

$$v_k''+[k^2-(2-\frac{m^2}{H^2})\frac{1}{\eta^2}]v_k=0$$

Let's consider first the very early times, $\eta^2\to+\infty$, (since $\eta\in(-\infty,0)$ when $t\in(-\infty,+\infty)$. Thus we can drop the $(2-\frac{m^2}{H^2})\frac{1}{\eta^2}$ term in this limit,since $k^2\gg\frac{1}{\eta^2}$ leaving us with:

$$v_k=\frac{1}{\sqrt{k}}e^{ik\eta},\qquad k|\eta|\gg 1 \tag{37}$$

we see that the mode functions take the form like in Minkowski space case. This means that the gravitational effect on the field modes is not significant.

At very late times the $(2-\frac{m^2}{H^2})\frac{1}{\eta^2}$ becomes dominating term and we neglect the $k^2$ part. In this limit the equation for the mode function becomes:

$$v_k''-(2-\frac{m^2}{H^2}\frac{1}{\eta^2})=0 \tag{38}$$

this the homogeneous equation which is called Cauchy-Euler equation and one can find its general solution as follows: let's change the variable $\eta \equiv e^t$ and calculate the second order derivative:

$$\frac{dv_k}{d\eta} = \frac{dv_k}{d(e^t)}\frac{d(e^t)}{d\eta} = e^{-t}\frac{dv_k}{dt}$$

$$\frac{d^2v_k}{d\eta^2} = e^{-t}\frac{d}{dt}(e^{-t}\frac{dv_k}{dt}) = e^{-2t}\frac{d^2v_k}{dt^2} - e^{-2t}\frac{dv_k}{dt}$$

if we plug in the original equation

$$[\frac{d^2v_k}{dt^2} - \frac{dv_k}{dt}]e^{-2t} - (2 - \frac{m^2}{H^2})e^{-2t}v_k = 0$$

multiplying by $e^{-2t}$ we get an equivalent equation

$$\frac{d^2v_k}{dt^2} - \frac{dv_k}{dt} - (2 - \frac{m^2}{H^2}) = 0$$

its characteristic equation is

$$n^2 - n - (2 - \frac{m^2}{H^2}) = 0$$

this has solutions

$$n_{1,2} = \frac{1}{2} \pm \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$$

since the $n_1$ and $n_2$ are different the equation has a general solution in the following form:

$$v_k(\eta) = A|\eta|^{n_1} + B|\eta|^{n_2}, \qquad k|\eta| \ll 1 \tag{39}$$

at late times $\eta \to 0$ the dominating term in the solution is one which has larger negative exponent:

$$v_k(\eta) \sim B|\eta|^{n_2}$$

We have seen that the asymptotics of the mode function are defined by the quantity $k|\eta|$. Let's focus on this for a while. A wave with wavenumber $\mathbf{k}$ has a comoving wavelength $L \sim k^{-1}$ and a corresponding physical length $L_p = a(t)L$. Thus one can write:

$$k|\eta| \sim \frac{1}{L}\frac{1}{aH} = \frac{H^{-1}}{L_p} \tag{40}$$

we used the definition of the conformal time, $|\eta| = \frac{1}{aH}$. Since $H^{-1}$ is a horizon scale one can interpret the quantity above as a measure for the modes being inside and beyond the horizon. Namely, large values of $k|\eta|$ at a given time $eta$ corresponds to the wavelenghts much shorter than the horizon scale $H^{-1}$. These modes are called subhorizon modes. Within the horizon scale the modes are not affected by the curvature. On the other hand, $k|\eta| \ll 1$ corresponds wavelengths which are much larger than the horizon scale and hence signifficantly affected by gravity. These are called superhorizon modes.

## 4 The Bunch-Davies vacuum

In a curved spacetime it is usually difficult to define a vacuum state. The main reason is that the Hamiltonian explicitly depends on time and hence energy is not conserved. Becuase of this one cannot define a vacuum state as a state with minimum energy. At some time it has the least energy but afterwards it doesn't. Therefore one defines the instantaneous vacuum state at fixed time $\eta = \eta_0$. But even in this prescription one faces with a imaginary frequency problem, i.e for some modes the frequency becomes imaginary. Because

$$\omega_k = k^2 + (\frac{m^2}{H^2} - 2)\frac{1}{\eta^2}$$

and $m \ll H$ there always exist such modes for which $k|\eta_0| \ll 1$ relation is satisfied and thus $\omega_k^2 < 0$. In this case the minimization of the energy becomes impossible. But for $k|\eta_0| \gtrsim 1$ modes the instantaneous vacuum is always defined. Hence instantaneous vacuum state cannot be defined for all modes. Now if we want to accommodate observationally relevant modes in order to satisfy $k|\eta_0| \gtrsim 1$ relation we need to choose relevant time point and this corresponds to the early times.

One of the main features of the de Sitter space is that it has a preferred vacuum state. Namely, at early times it has such a vacuum state that is defined basically like a vacuum state in Minkowski space for all modes. At early times where $\frac{1}{\eta^2}$ is negligible the frequency becomes almost constant. Hence $\zeta_k$ modes has a strong adiabatic regime. This means that the modes are not essentially affected by gravity at early times. Therefore one defines a vacuum state like in Minkowski space for all modes. Analytically this can be expressed by the following relations:

$$v_k(\eta) \to \frac{1}{\sqrt{\omega_k}} e^{i\omega_k \eta}, \qquad \frac{v_k'(\eta)}{v_k(\eta)} \to i\omega_k \tag{41}$$

when $\eta \to -\infty$. The second relation is a condition for the adiabatic regime. The mode functions which satisfy these relations are called the Bunch-Davies vacuum. The mode functions for the Bunch-Davies vacuum in terms of the Bessel functions is given in the following form:

$$v_k(\eta) = \sqrt{\frac{\pi|\eta|}{2}}[J_n(k|\eta|) - iY_n(k|\eta|)] \tag{42}$$

Since the de Sitter spacetime describes some stage of the inflation the theory we have developed is valued only for some period of time. Let's denote the earliest time as $\eta_i$. Also we don't consider the superhorizon modes at $\eta_i$, namely $k|eta_i| < 2$. The adiabatic regime for a $\zeta_k$ modes which were subhorizon at $\eta_i$ is valid until the time at which those modes cross the horizon $\eta_k$ i.e until $k\eta_k \sim 1$ is satisfied. Thus in the $\eta_i < \eta < \eta_k$ interval the Bunch-Davies mode function coincides with Minkowski one. We will consider the cases inside and outside the interval.

$$v_k(\eta)\frac{1}{\sqrt{k}}e^{ik\eta}, \qquad \eta_i < \eta < \eta_k \tag{43}$$

At boundaries the different asymptotics need to match. Taking this into account for (39) and (43) expressions at $\eta = \eta_k$ we get

$$v_k(\eta) = A_k\frac{1}{\sqrt{k}}|\frac{\eta}{\eta_k}|^{n_1} + B_k\frac{1}{\sqrt{k}}|\frac{\eta}{\eta_k}|^{n_2}, \tag{44}$$

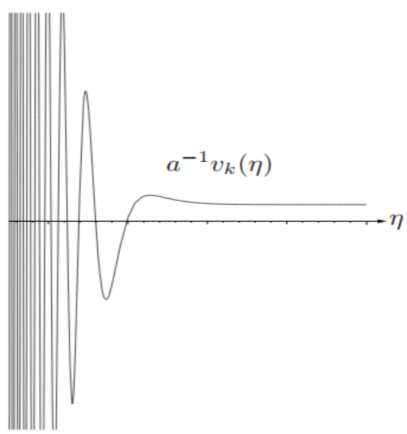
for the $\eta > \eta_k$ times. Now the interesting feature of the Bunch-Davies mode function appears when one applies the (44) solution for the superhorizon modes i.e $k|\eta| \ll 1$. Since in this case $|\eta| \ll k^{-1}$ the term with positive exponent i.e $n_1$ becomes negligible. Then the (44) reduce to

$$v_k \sim \frac{1}{\sqrt{k}}|\frac{\eta}{\eta_k}|^{\frac{1}{2}-n} \tag{45}$$

where $n = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}}$. Let's remind us that the $\zeta$ field was actually an auxiliary field i.e $\zeta \equiv a(\eta)\phi$. Then the mode function for the $\phi$ field is defined as:

$$a^{-1}v_k(\eta) \propto |\eta|v_k(\eta) \propto |\eta|^{\frac{3}{2}-n} \tag{46}$$

as was mentioned before in most cosmological circumstances $m \ll H$ hence $n \approx \frac{3}{2}$. Thus the mode function (46) becomes almost scale invariant(constant) at late times. This behaviour of the mode function can be plotted for the massless case in the following way: (In the firgure below the imaginary part of the Bunch-Davies mode function is plotted. The time axes evolves in the logarithmic scale.)




# 5 Quantum fluctuations

We saw that there exist such situations where the frequency can be imaginary and hence in this case the particle interpretation doesn't make any sense. Therefore as an observable parameter the fluctuation amplitude or power spectrum becomes useful observable. Because even though that the frequency is imaginary the superhorizon modes are real and hence fluctuations defined in terms of correlation functions is more fundamental quantities.

One of the possibilities of characterizing the quantum fluctuation amplitude is a equal time correlation function for an arbitrary state $|\psi\rangle$, $\langle\psi|\,\zeta(\vec{x},\eta),\zeta(\vec{y},\eta)\,|\psi\rangle$ . The correlation function for an vacuum state is given as:

$$\langle 0|\,\zeta(\vec{x},\eta),\zeta(\vec{y},\eta)\,|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|v_k(\eta)|^2 e^{i\vec{k}(\vec{x}-\vec{y})} = \frac{1}{4(\pi)^2}\int k^2 dk |v_k(\eta)|^2 \frac{sinkL}{kL} \tag{47}$$

where $L = |\vec{x}-\vec{y}|$ is the comoving distance between co-ordinates. The wavenumbers $k \sim \frac{1}{L}$ dominate in the integral. Thus, the magnitude of the correlation function is as:

$$\langle 0|\,\zeta(\vec{x},\eta),\zeta(\vec{y},\eta)\,|0\rangle \sim k^3|v_k|^2$$

Another possibility to characterize the quantum fluctuations is using the field which is averaged over some space domain.

$$\delta\zeta_L^2(\eta) \equiv \langle\psi|\,[\hat{\zeta}_L(\eta)]^2\,|\psi\rangle$$

here, the $\hat{\zeta}_L(\eta)$ is a field averaged over distance L. The averaging procedure over an arbitrary domain is performed by introducing a distribution function called window function. A window function for scale L is a function W(x) which is of order 1 if $|x| \lesssim L$ and rapidly decays if $|x| \gg L$. It satisfies the normalization condition:

$$\int W(\vec{x})d^3x = 1$$

Then we can calculate the averaged field by multiplying it with the window function and integrate:

$$\hat{\zeta}_L(\eta) = \int d^3x \hat{\zeta}(\vec{x},\eta) W_L(\vec{x})$$

where, $W_L(x) = \frac{1}{L^3}W(\frac{x}{L})$. The convenient way to calculate this is to use the Fourier image of the window function defined as

$$\int d^3x W_L(\vec{x}) e^{-i\vec{k}\vec{x}} = w(kL)$$

. Then the amplitude will be

$$\langle 0|\,[(\int d^3x \hat{\zeta}(\vec{x},\eta) W_L(\vec{x})]^2\,|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|v_k|^2 |w(\vec{k}L)|^2$$

By definition of the window function only $k \lesssim L^{-1}$ modes contribute. Thus we integrate up to $L^{-1}$ value

$$\int \frac{d^3k}{(2\pi)^3}\frac{1}{2}|v_k|^2|w(\vec{k}L)|^2 \sim \int_0^{L^{-1}} dk k^2 |v_k|^2 \sim \frac{1}{L^3}|v_k|^2$$

Finally, we get the expression for the amplitude

$$\delta\zeta_L^2(\eta) \sim k^3|v_k|^2 \tag{48}$$

We see that the two results -one from equal time correlation function and from the averaged field coincide. Actually, the square root of this amplitude,

$$\delta\zeta_L(\eta) \sim k^{3/2}|v_k|$$

is called the amplitude of fluctuations on scale L. It is defined up to some factor of order 1.

*Quantum fluctuations in Bunch-Davies vacuum.*
Now let's apply the formalism developed in the previous section to the Bunch-Davies vacuum. So far we have worked with the auxiliary field $\zeta$ which was defined as $\zeta \equiv a(\eta)\phi$. Now we return to the original field $\phi$. Then the amplitude of quantum fluctuations is as:

$$\delta\phi_L(\eta) = a^{-1}(\eta)k^{3/2}|v_k(\eta)| \tag{49}$$

if remind us the relation between scale factor and the fixed Hubble constant

$$a(\eta) = -\frac{1}{H\eta}$$

(- sign comes from the fact $\eta \in (-\infty, 0)$) and substitute this in expression above we get

$$\delta\phi_L(\eta) = -H\eta k^{3/2}|v_k(\eta)| \tag{50}$$

or

$$\delta\phi_L(\eta) = H|\eta|k^{\frac{3}{2}}|v_k(\eta)| \tag{51}$$

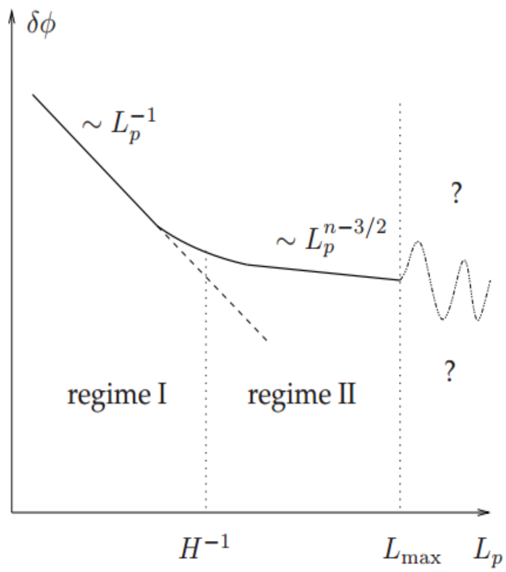
Since the two point correlation function gives the amplitude of fluctuations,

$$\langle 0|\,\phi(\vec{x},\eta),\phi(\vec{y},\eta)\,|0\rangle = \frac{H^2\eta^2}{L^3}|v_k(\eta)|^2 \tag{52}$$

If we plug the asymptotic expressions of the Bunch-Davies mode function in this equation and express it in real space(in terms of $L_p = a(t)L$) we get different values for the amplitude of fluctuation in different regions.

$$\delta\phi_{L_p} = \begin{cases} undefined, \eta < \eta_i or L_p \gtrsim H^{-1}\frac{\eta_i}{\eta} \\ L_p^- 1, L_p < H^{-1} \\ H|L_pH|^{n-\frac{3}{2}}, L_pH^{-1} \end{cases} \tag{53}$$

at fixed time this can be plotted in the following way:




In the figure above the dashed line corresponds to fluctuations for Minkowski mode function. We see that in smaller than horizon scale distances the fluctuation amplitude coincide with the Minkowski one. But in the superhorizon region ($H^{-1} < L_p < L_{max}$) it becomes constant but larger compared to fluctuations in Minkowski space. This is due to gravitational effect. Gravity enhances the fluctuations. In the distances larger than $L_{max}$ the quantum state of the field is undefined. Therefore at late times fluctuations at cosmologically relevant scales are independent of the initial conditions at $\eta = \eta_i$ and coincide with the fluctuations in the Bunch-Davies vacuum state. Thus the expansion of the de Sitter space makes any initial state coincide with the Bunch-Davies vacuum state and this is the main advantage of the de Sitter space. The enhancement in the fluctuations as was shown in the figure above is used in cosmology to explain the formation of large structures like galaxies, galaxy clusters in the observable universe.

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