Processes driving (P)reheating

Report relating to a talk given on December 2nd 2016
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I Introduction

This is a brief review which aims to describe the central processes behind reheating and preheating.
At the end of inflation, the universe is empty and cold. Solely the homogeneous zero mode of the inflaton field comprises energy. Exponential expansion has diluted all other fields. For the present universe to arise from this and hot Big Bang theory to work, the energy has to be transfered from the inflaton field to Standard model degrees of freedom. The universe has to *reheat*.
The structure of this report is as follows. In chapter II, we will concisely summarise the concepts of what one might call the "old" theory of reheating which is based on the *perturbative* decay of inflaton quanta. In chapter III, we will focus on the highly relevant *non perturbative* effects in the form of parametric resonance, termed *preheating*. This chapter is seperated in the regime of *narrow* and *broad* resonance with the main emphasis being found on the latter.
Whereas this work is mainly based on [1], [2], [3], [4] all provide very conducive descriptions of the theory.

II Elementary theory of Reheating

At the end of inflation, the inflaton field ϕ oscillates around the minimum of its potential V and obeys the Klein-Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{II.1}$$

where $H = \frac{\dot{a}}{a}$ denotes the Hubble parameter. Its oscillatory solution is

$$\phi \simeq \Phi(t)\sin(mt), \qquad \text{where} \qquad \Phi \sim t^{-1}. \tag{II.2}$$

We now introduce the model of a coupling to another type of scalar particle[1] χ

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) - \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{2}g^2\phi^2\chi^2. \tag{II.3}$$

Expanding the potential around its minimum at $\phi = \sigma$

$$V(\phi) = \frac{1}{2}m^2(\phi-\sigma)^2 + \mathcal{O}\left(\phi^4\right) \tag{II.4}$$

we shift ϕ by its vacuum expectation value σ, i.e. $\phi \to \phi + \sigma$, such that

$$V(\phi) + \frac{1}{2}g^2\phi^2\chi^2 \to \frac{1}{2}m^2\phi^2 + g\sigma\phi\chi^2 + \frac{1}{2}g^2\phi^2\chi^2. \tag{II.5}$$

As a result, the Lagrangian density contains the interaction part

$$\mathcal{L} \supset g\sigma\phi\chi^2 + \frac{1}{2}g^2\phi^2\chi^2. \tag{II.6}$$

Then, corresponding decay rates can be established via Quantum Field Theory, such as

$$\Gamma_{\phi\to\chi\chi} = \frac{g^4\sigma^2}{8\pi m}. \tag{II.7}$$

The resulting damping of the inflaton oscillations can find a phenomenological description by introducing an additional friction term $\Gamma_\phi\phi$ in (II.1):

$$\ddot{\phi} + \left(3H + \Gamma_\phi\right)\dot{\phi} + m^2\phi = 0. \tag{II.8}$$

[1] The introduction of a coupling to a fermion via a Yukawa coupling $h\phi\psi\bar{\psi}$, which results in a corresponding decay rate, would be a possibility just as well. Nevertheless, this is not explicitely performed as it does not provide us with any qualitatively different features. Moreover, it shows to typically only being significant at very late stadia of reheating, [1]

Here, Γ_ϕ comprises the contribution of all decay modes relevant for the decay of the inflaton. This alters the solution (II.2) by a further decrease of the amplitude

$$\Phi \sim t^{-1} \mathrm{e}^{-\frac{1}{2}\Gamma_\phi t}. \tag{II.9}$$

The time evolution of the number density of ϕ particles and the respective energy density can be written as

$$\begin{aligned} \frac{d}{dt}\left(n_\phi a^3\right) &= -\Gamma_\phi n_\phi a^3 \\ \frac{d}{dt}\left(\rho_\phi a^3\right) &= -\Gamma_\phi \rho_\phi a^3. \end{aligned} \tag{II.10}$$

From this, we can see that the total comoving energy and comoving number of inflaton particles is initially conserved as $\Gamma_\phi \ll H$. Once $H \sim \frac{2}{3t}$ decreased sufficiently, such that $\Gamma_\phi > H$, the inflaton quickly decays into χ particles within its lifetime, i. e. in less than a Hubble time: $\tau_\phi = \Gamma_\phi^{-1} < H^{-1}$. Based on this consideration and assuming a thermodynamic equilibrium to establish quickly, it is now possible to give an estimate for the crucial reheating temperature at the time of reheating $t_r \sim \frac{2}{3}\Gamma_\phi^{-1}$. Equating the energy densitiy of the universe at this time[2]

$$\rho(t_r) = \frac{3\Gamma_\phi^2 M_p^2}{8\pi}. \tag{II.11}$$

with the energy density of a plasma of relativistic particles

$$\rho(T_r) = \frac{\pi}{30} g_*(T_r)\, T_r^4 \tag{II.12}$$

where g_* denotes the number of relativistic degrees of freedom, we find

$$T_r \simeq 0.2 \left(\frac{100}{g_*}\right)^{\frac{1}{4}} \sqrt{\Gamma_\phi M_p}. \tag{II.13}$$

To obtain an approximate value for T_r, one can estimate g_* as $10^2 - 10^3$ and employ constraints on the couplings, e.g. $g \lesssim 10^{-3}$, and the mass of the inflaton m_ϕ in our model. On that account, we require the anisotropies of the CMB not to be created excessively and radiative corrections to enforce the flatness of the potential. This yields a result of $T_r \lesssim 10^{11}$GeV.
On this occasion, we shall briefly return to the choice of the considered decay rates. We solely used the explicit statement of the constant rate (II.7) for the decay $\chi \to \chi\chi$ in our above line of argumentation as there exists a caveat with $\Gamma_{\phi\phi\to\chi\chi} = \frac{g^4\Phi^2}{8\pi m}$. In this scenario, the decay rate $\Gamma_{\phi\phi\to\chi\chi} \sim t^{-2}$ decreases faster with increasing time than the Hubble parameter $H \sim t^{-1}$. Thus, our aforementioned condition $\Gamma_\phi > H$ might never be fulfilled. In this instance, one has to demand spontaneous symmetry breaking, i.e. $\sigma \neq 0$, or an appropriate coupling of the inflaton field to fermions for reheating to be effective on the level of the elementary theory presented in this chapter.

[2] A.D. Linde, *Particle Physics and Inflationary Cosmology*, [5]

III Preheating by Parametric Resonance

In the previous chapter we performed a *perturbative* analysis of reheating. That is, we treated the inflaton field as inflaton quanta decaying individually with their corresponding, perturbative, decay rates being described by usual methods of Quantum field theory. This approach falls short of appreciating the nature of the inflaton field which, at the end of inflation, presents itself as a coherently oscillating homogeneous field. In the following chapter, we will therefore consider the production of χ particles induced by the time dependant background field ϕ,. This process, due to it usually being relevant prior to the elementary theory presented above, was termed *preheating*. As we will see, it embodies a conceptually *non-perturbative* effect.
Hence, let us now examine the quantum field $\hat{\chi}$ with the background ϕ

$$\hat{\chi}(t,x) = \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}} \left(\hat{a}_k \chi_k(t) \mathrm{e}^{-ikx} + \hat{a}_k^\dagger \chi_k^\dagger(t) \mathrm{e}^{ikx} \right) \tag{III.1}$$

where $\hat{a}_k^\dagger$ and $\hat{a}_k$ denote creation and annihilation operators while the mode functions obey the equation

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + m_\chi^2 + g^2\phi^2(t) \right) \chi_k = 0 \tag{III.2}$$

Here, the term in parentheses is the time-dependant frequency ω_k, which can be interpreted to comprise a time-dependant mass. Defining the dimensionless time $z = mt$ and neglecting the effects of expansion for the moment being, i.e. $a = \text{const.}$, the mode function equation (III.2) can be rewritten as the *Mathieu equation*

$$\chi_k'' + (A_k - 2q\cos(2z))\chi_k = 0\,, \tag{III.3}$$

with

$$A_k = \frac{k^2/a^2 + m_\chi^2}{m^2} + 2q\,, \quad q = \frac{g^2\Phi^2}{4m^2}\,. \tag{III.4}$$

Here, we have used the trigonometric identity $\cos 2x = 1 - 2\sin^2 x$ and primes denote derivatives with respect to z. The crucial trait of the solutions of the Mathieu equation is that they can be classified as stable and instable for certain regions of the parameters A_k and q. Figure III.1 shows a graph depicting this characteristic. Employing *Floquet's theorem*, these solutions within certain bands Δk of instability take the form

$$\chi_k = p(z)\,\mathrm{e}^{\mu_k z} \tag{III.5}$$

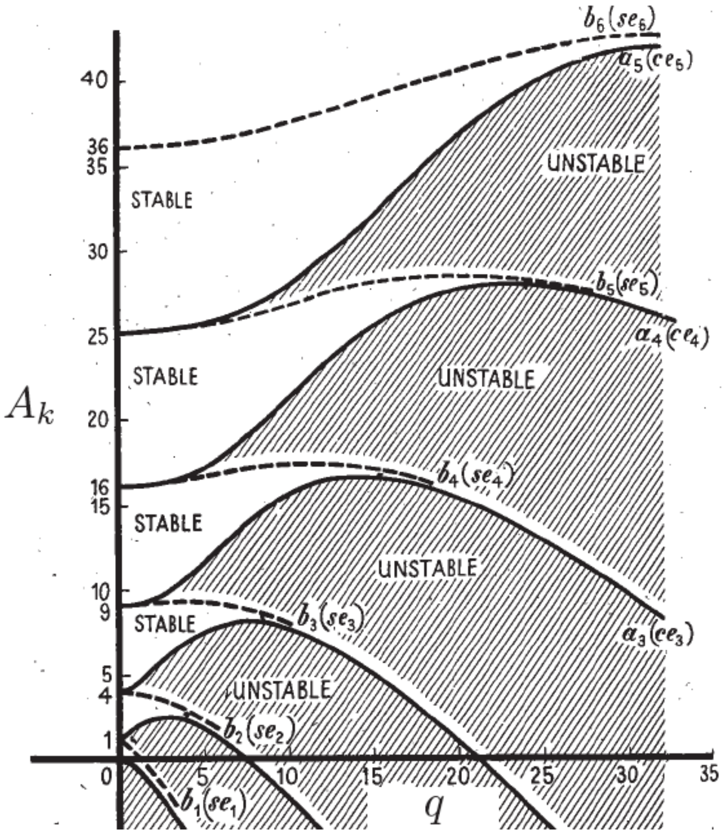

Figure III.1: Chart of stability and instability bands of the Mathieu euqation dependant on the parameters q and A_k; reproduced from [3], [6]

with μ_k being the Floquet index and $p(z)$ a function periodic in z.
The occupation number of a certain mode can be estimated as the energy of that very mode divided by the energy ω_k of each particle:

$$n_k = \frac{\omega_k}{2}\left(\frac{\left|\dot{\chi}_k^2\right|}{\omega_k^2} + \left|\chi_k^2\right|\right) - \frac{1}{2}. \tag{III.6}$$

Hence, (III.5) will result in an exponential growth of the occupation number

$$n_k \sim \mathrm{e}^{2\mu_k z}, \tag{III.7}$$

which is identified with particle production.

III.1 Narrow Resonance

We will now focus on the regime of $q \ll 1$. In this case, resonant solutions are found in narrow bands near $A_k^{(n)} \simeq n^2$ with $n \in \mathbb{N}$, cf figure III.1. Their width in momentum space

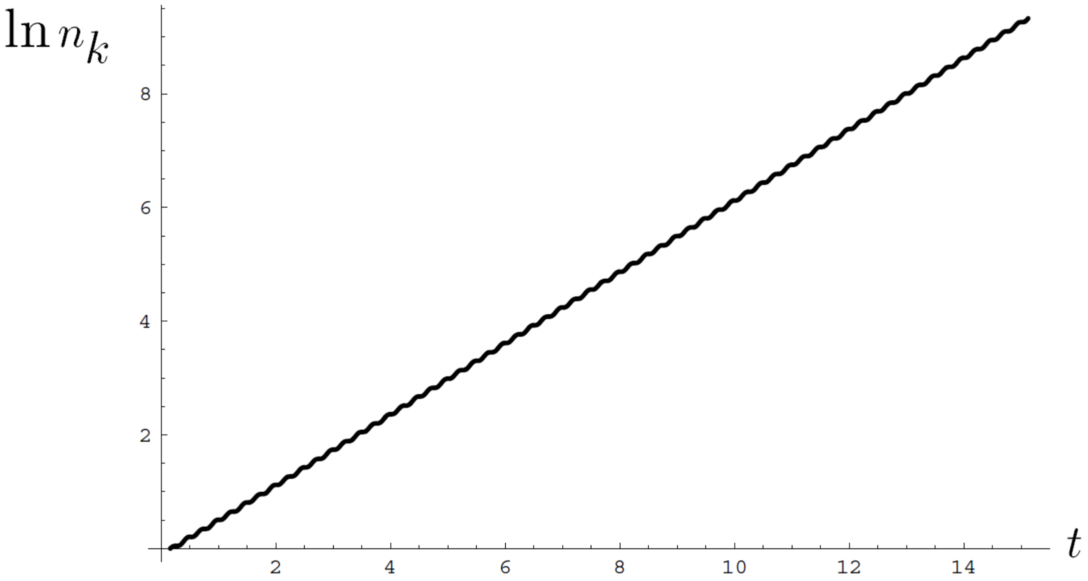

Figure III.2: Narrow parametric resonance for the field χ in Minkowski space for $q \sim 0.1$ as obtained by numerical methods, t is depicted in units of $2\pi/m$; reproduced from [1] by *L. Kofman, A. Linde, A. Starobinsky*, 1997

scales as $\Delta k \sim mq^n$. The widest and most important band is the first one situated at

$$k = m\left(1 \pm \frac{q}{2}\right) \tag{III.8}$$

For simplicity of the expression given, m_χ is assumed to be negligible. The corresponding Floquet index

$$\mu_k = \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{k}{m} - 1\right)^2} \tag{III.9}$$

is maximal at the center of the resonance band and vanishes at its edges. As a result, exponential particle production will arise, although solely in very narrow bands. Figure III.2 shows the result of numerical calculations for the regime of narrow resonance under the neglection of expansion. It should not remain unmentioned that the effect of narrow resonance can also be treated as Bose condensation of χ particles, [7].
The inclusion of expansion in our considerations will affect this process twofold. Firstly, $\Phi(t)$ will decrease with increasing time, therefore reducing q and with it narrowing the bands even further. Secondly, the momenta $k \to k/a$ will experience redshitfting. As a consequence of this, modes which are within one of the narrow bands at a given time, will be swiftly shifted away from it.

III.2 Broad Resonance

In this section, we will turn to the regime of $q \gg 1$, i.e. that of broad parametric resonance.
Against the backdrop of WKB theory, it is valuable to seperately investigate the mode function equation (III.2) in the case in which ω_k is varying rapidly and in which it is not. In the latter, the solutions for χ_k will not grow and therefore, no significant particle creation will occur. If, on the contrary, w_k does experience rapid change with time, WKB ceases to be applicable which leads us to demanding the condition

$$\mathcal{A} := \left| \frac{\dot{\omega}_k}{\omega_k^2} \right| \gg 1 \tag{III.10}$$

to be fulfilled for particle creation. (III.10) is known as the *non-adiabaticity condition* since the particle number n_k is an adiabatic invariant which is constant over time for $|\mathcal{A}| \ll 1$. For small wavenumbers k and small m_χ, one finds

$$\mathcal{A} \simeq \frac{\dot{\phi}}{g\phi^2} \tag{III.11}$$

Hence, in this case[1], $\mathcal{A}$ diverges for

$$\phi(t) \sim 0 \tag{III.12}$$

which ocurs twice every oscillation of the inflaton field at the points in time

$$t_j = j\frac{\pi}{m}, \;\; j \in \mathbb{Z} \quad \text{where} \quad \phi\left(t_j\right) = 0. \tag{III.13}$$

To incorporate the expansion of the universe in our study of broad parametric resonance it proves to be beneficial to introduce the function

$$X_k(t) = a^{\frac{3}{2}} \chi_k(t). \tag{III.14}$$

Hereby, we can absorb the Hubble friction term as part of the mode function equation and rewrite (III.2) in the simpler form of

$$\ddot{X} + \omega_k^2 X_k = 0 \tag{III.15}$$

where

$$\omega_k^2 = \frac{k^2}{a^2} + m_\chi^2 + g^2\phi^2 + \delta \tag{III.16}$$

and

$$\delta = -\frac{3}{4}\left(\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a}\right) = -\frac{3}{4}\left(3H^2 + 2\dot{H}\right). \tag{III.17}$$

[1] In the case of finite k and m_χ, one generally finds $R = \frac{g^2\phi\dot{\phi}}{\left(\frac{k}{a^2} + m_\chi^2 + g^2\phi^2\right)^{\frac{3}{2}}} > 1$, which in turn provides a limit on the range of k effectively taking part in the resonance.

As the term δ proves to be negligible for $a \sim t^{2/3}$, it will not be regarded in the following analysis.
Away from the times t_j, the solutions $X_k(t)$ evolve adiabatically as

$$X_k^j(t) = \frac{\alpha_k^j}{\sqrt{2\omega_k}} e^{-\mathrm{i}\int^t \omega_k dt} + \frac{\beta_k^j}{\sqrt{2\omega_k}} e^{\mathrm{i}\int^t \omega_k dt} \tag{III.18}$$

with $\alpha_k^j, \beta_k^j = \text{const.}$ between t_{j-1} and t_j and accordingly

$$X_k^{j+1}(t) = \frac{\alpha_k^{j+1}}{\sqrt{2\omega_k}} e^{-\mathrm{i}\int^t \omega_k dt} + \frac{\beta_k^{j+1}}{\sqrt{2\omega_k}} e^{\mathrm{i}\int^t \omega_k dt}. \tag{III.19}$$

Here, the Bogoliubov coefficients α_k and β_k fulfil the initial and normalisation conditions $\alpha_k(t \to 0) = 1$, $\beta_k(t \to 0) = 0$, $|\alpha_k|^2 - |\alpha_k|^2 = 1$. To focus on the instances of particle production, i.e. around $\phi \sim 0$, we introduce a linearisation of $\phi(t) \simeq \Phi m\left(t - t_j\right)$ around t_j to obtain

$$\ddot{X} + \left(\frac{k^2}{a^2} + m_\chi^2 + g^2 m^2 \Phi^2 \left(t - t_j\right)^2\right) X_k = 0 \tag{III.20}$$

By defining the rescaled momentum and time variables

$$\kappa := \frac{k^2/a^2 + m_\chi^2}{mg\Phi}, \qquad \tau := \sqrt{mg\Phi}\left(t - t_j\right), \tag{III.21}$$

the mode function equation with linearised ϕ (III.20) can now be rescaled to

$$\frac{\mathrm{d}^2 X_k}{\mathrm{d}\tau^2} + \left(\kappa^2 + \tau^2\right) X_k = 0. \tag{III.22}$$

Hence, our problem of parametric resonance has been reduced to the known quantum mechanical problem of one-dimensional Schrödinger scattering. In particular, (III.22) describes scattering at an inverted parabolic potential $V = -\tau^2$ which is solved by linear combinations of parabolic cylinder functions $W\left(-\frac{\kappa^2}{2}; \pm\sqrt{2}\tau\right)$. In essence, (III.18) and (III.19) describe asymptotic solutions for incoming $\left(t < t_j\right)$ and outgoing waves $\left(t > t_j\right)$ of scattering taking place at t_j. Consequently, incoming and outgoing amplitudes, α_k^j, β_k^j and $\alpha_k^{j+1}, \beta_k^{j+1}$ respectively, can be linked by the reflection and transmission amplitudes R_k and T_k:

$$\begin{pmatrix} \alpha_k^{j+1} e^{-\mathrm{i}\vartheta_k^j} \\ \beta_k^{j+1} e^{\mathrm{i}\vartheta_k^j} \end{pmatrix} = \begin{pmatrix} \frac{1}{T_k} & \frac{R_k^*}{T_k^*} \\ \frac{R_k}{T_k} & \frac{1}{T_k^*} \end{pmatrix} \begin{pmatrix} \alpha_k^j e^{-\mathrm{i}\vartheta_k^j} \\ \beta_k^j e^{\mathrm{i}\vartheta_k^j} \end{pmatrix} \tag{III.23}$$

where we introduced

$$\vartheta_k^j = \int_0^t \omega_k(t)\, dt. \tag{III.24}$$

The amplitudes in question are found to be

$$R_k = -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi_k}}{\sqrt{1+\mathrm{e}^{\pi\kappa^2}}}\,, \qquad T_k = -\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}\varphi_k}}{\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}} \tag{III.25}$$

with

$$\varphi_k = \arg\Gamma\left(\frac{1+\mathrm{i}\kappa^2}{2}\right) + \frac{\kappa^2}{2}\left(1+\ln\frac{2}{\kappa^2}\right). \tag{III.26}$$

Additionally, from (III.25), we can convince ourselves that the reflection and transmission amplitudes fulfil the relation

$$|R_k|^2 + |T_k|^2 = 1\,. \tag{III.27}$$

Then, (III.23) can be solved for the coefficients α_k^{j+1} and β_k^{j+1}:

$$\begin{aligned}\begin{pmatrix}\alpha_k^{j+1}\\ \beta_k^{j+1}\end{pmatrix} &= \begin{pmatrix}\frac{1}{T_k} & \mathrm{e}^{2\mathrm{i}\vartheta_k^j}\frac{R_k^*}{T_k^*}\\ \mathrm{e}^{-2\mathrm{i}\vartheta_k^j}\frac{R_k}{T_k} & \frac{1}{T_k^*}\end{pmatrix}\begin{pmatrix}\alpha_k^j\\ \beta_k^j\end{pmatrix}\\ &= \begin{pmatrix}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\mathrm{e}^{\mathrm{i}\varphi_k} & \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2+2\mathrm{i}\vartheta_k^j}\\ -\mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2-2\mathrm{i}\vartheta_k^j} & \sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\mathrm{e}^{-\mathrm{i}\varphi_k}\end{pmatrix}\begin{pmatrix}\alpha_k^j\\ \beta_k^j\end{pmatrix}.\end{aligned} \tag{III.28}$$

Employing the relation of the Bogoliubov coefficients and the corresponding occupation numbers[2]

$$n_k^j = \left|\beta_k^j\right|^2\,, \qquad \left|\alpha_k^j\right|\left|\beta_k^j\right| = n_k^j\left(n_k^j+1\right), \tag{III.29}$$

we arrive at the number density of outgoing particles n_k^{j+1} in a mode as a consequence of a single scattering process dependant on the number density of incoming particles n_k^j

$$\begin{aligned} n_k^{j+1} &= \mathrm{e}^{-\pi\kappa^2}\left|\alpha_k^j\right|^2 + \left(1+\mathrm{e}^{-\pi\kappa^2}\right)\left|\beta_k^j\right|^2\\ &\quad + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2+2\mathrm{i}\vartheta_k^j}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\mathrm{e}^{-\mathrm{i}\varphi_k}\alpha_k^{j*}\beta_k^j - \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2-2\mathrm{i}\vartheta_k^j}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\mathrm{e}^{\mathrm{i}\varphi_k}\alpha_k^j\beta_k^{j*}\\ &= \mathrm{e}^{-\pi\kappa^2}\left|\alpha_k^j\right|^2 + \left(1+\mathrm{e}^{-\pi\kappa^2}\right)\left|\beta_k^j\right|^2\\ &\quad + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\left(\mathrm{e}^{\mathrm{i}\left(2\vartheta_k^j-\varphi_k\right)}\alpha_k^{j*}\beta_k^j - \mathrm{e}^{-\mathrm{i}\left(2\vartheta_k^j-\varphi_k\right)}\alpha_k^j\beta_k^{j*}\right)\\ &= \mathrm{e}^{-\pi\kappa^2}\left|\alpha_k^j\right|^2 + \left(1+\mathrm{e}^{-\pi\kappa^2}\right)\left|\beta_k^j\right|^2\\ &\quad + \mathrm{i}\mathrm{e}^{-\frac{\pi}{2}\kappa^2}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\left|\alpha_k^j\right|\left|\beta_k^j\right|\left(\mathrm{e}^{\mathrm{i}\left(2\vartheta_k^j-\varphi_k\right)}\mathrm{e}^{\mathrm{i}\left(\arg\beta_k^j-\arg\alpha_k^j\right)} - \mathrm{e}^{-\mathrm{i}\left(2\vartheta_k^j-\varphi_k\right)}\mathrm{e}^{-\mathrm{i}\left(\arg\beta_k^j-\arg\alpha_k^j\right)}\right)\\ &= \mathrm{e}^{-\pi\kappa^2} + \left(1+2\mathrm{e}^{-\pi\kappa^2}\right)n_k^j - 2\mathrm{e}^{-\frac{\pi}{2}\kappa^2}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\sqrt{n_k^j\left(n_k^j+1\right)}\sin\vartheta_{tot}^j \end{aligned} \tag{III.30}$$

[2] V. F. Mukhanov and Sergei Winitzki, *Introduction to quantum effects in gravity*, [8]

where we defined

$$\vartheta_{tot}^{j} = 2\vartheta_{k}^{j} - \varphi_{k} + \arg\beta_{k}^{j} - \arg\alpha_{k}^{j}. \qquad \text{(III.31)}$$

We shall now analyse the characteristic properties of our central result that is given by (III.30).
It describes a *step-like* creation of particles, cf figure III.3, While the occupation number is constant between two successive scatterings at t_j and t_{j+1} in consequence of α_k^j and β_k^j being constant, it changes at the points in time t_j. In the case of a vanishing occupation number $n_k^j = 0$, the first step is taken by $n_k^{j+1} = \mathrm{e}^{-\pi\kappa^2}$.
In order for particle creation to be effective, one has to require $\pi\kappa^2 \lesssim 1$ as exponential suppression will occur otherwise. Combining this condition with the definition of κ in (III.21), we find that high momenta will not be populated which will therefore called *infrared effect*. This allows for an estimate for the width of the resonance band as

$$\frac{k^2}{a^2} \lesssim \frac{gm\Phi}{\pi} - m_\chi. \qquad \text{(III.32)}$$

Given large occupation numbers, $n_k^j \gg 1$, we will encounter *exponential production of bosons* as (III.30) then takes the form

$$n_k^{j+1} \simeq \mathrm{e}^{2\pi\mu_k^j} n_k^j \qquad \text{(III.33)}$$

with the Floquet index

$$\mu_k^j = \frac{1}{2\pi}\ln\left(1 + 2\mathrm{e}^{-\pi\kappa^2} - 2\mathrm{e}^{-\frac{\pi}{2}\kappa^2}\sqrt{1+\mathrm{e}^{-\pi\kappa^2}}\, sin\vartheta_{tot}^j\right). \qquad \text{(III.34)}$$

Moreover, we can convince ourselves of the result (III.30) actually being *non-perturbative* as $\mathrm{e}^{1/g}$ is a non-analytic function at $g = 0$. Thus, the above results for parametric resonance cannot be obtained by a perturbative approach.
In addition, (III.30) allows the production of particles with *higher mass than that of the inflaton* itself since

$$m^2 < m_\chi^2 \ll gm\Phi \Rightarrow \kappa^2 \lesssim \pi^{-1} \qquad \text{(III.35)}$$

is a possible arrangement for significant particle production.
Furthermore, (III.30) gives rise to *resonant production*. In our derivation, we incorporated the occurrences of scattering to be periodic. It is the phase correlation between successive scatterings which determines the resulting particle production. Whereas the first two terms of (III.30) generally increase particle numbers in the manner of spontaneous particle creation, the last term may either contribute to an increase or a decrease of particle numbers and conceptually embodies induced particle creation. The Floquet index will be maximal for $\sin\vartheta_{tot} = -1$ and minimal, particularly it may assume negative values, for $\sin\vartheta_{tot} = 1$. In the framework of neglected expansion with a background field of constant amplitude, $\Phi(t) = $ const. and $a = $ const., ϑ_k^j is solely dependant on k. This generates a characteristic *structure of stability/instability k-bands*.

Taking expansion into account, the phase will instead additionally present a time-dependence. In consequence, this interplay of phase and expansion will lead to a succession of particle number increases and decreases with a stochastic growth of particle numbers at roughly 3/4 of the scattering instances. Hence, this process is known as *stochastic preheating*, cf figure III.4.
Eventually, we will turn to estimating the net particle creation as the result of scattering processes up to the instance t_j.

$$n_k(t) = \frac{1}{2} \mathrm{e}^{2\pi \sum_j \mu_k^j} = \frac{1}{2} \mathrm{e}^{2m\mu_k^{eff} t} \tag{III.36}$$

where the effective Floquet index

$$\mu_k^{eff} = \frac{\pi}{mt} \sum_j \mu_k^j \tag{III.37}$$

was introduced. This allows to infer the total number density of particles created. We finally arrive at

$$n_\chi(t) = \frac{1}{(2\pi a)^3} \int d^3k \; n_k(t) = \frac{1}{4\pi^2 a^3} \int dk \; k^2 \mathrm{e}^{2\pi\mu_k^{eff} t}. \tag{III.38}$$

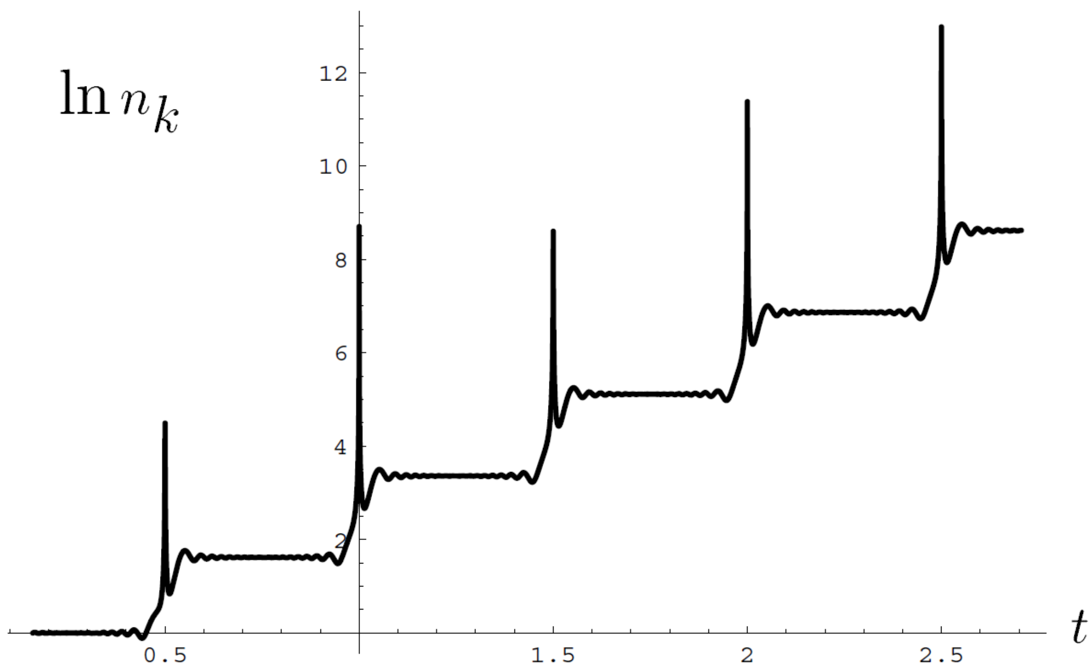

Figure III.3: Broad parametric resonance for the field χ in Minkowski space for $q \sim 2 \cdot 10^2$ as obtained by numerical methods, t is depicted in units of $2\pi/m$; reproduced from [1] by *L. Kofman, A. Linde, A. Starobinsky*, 1997

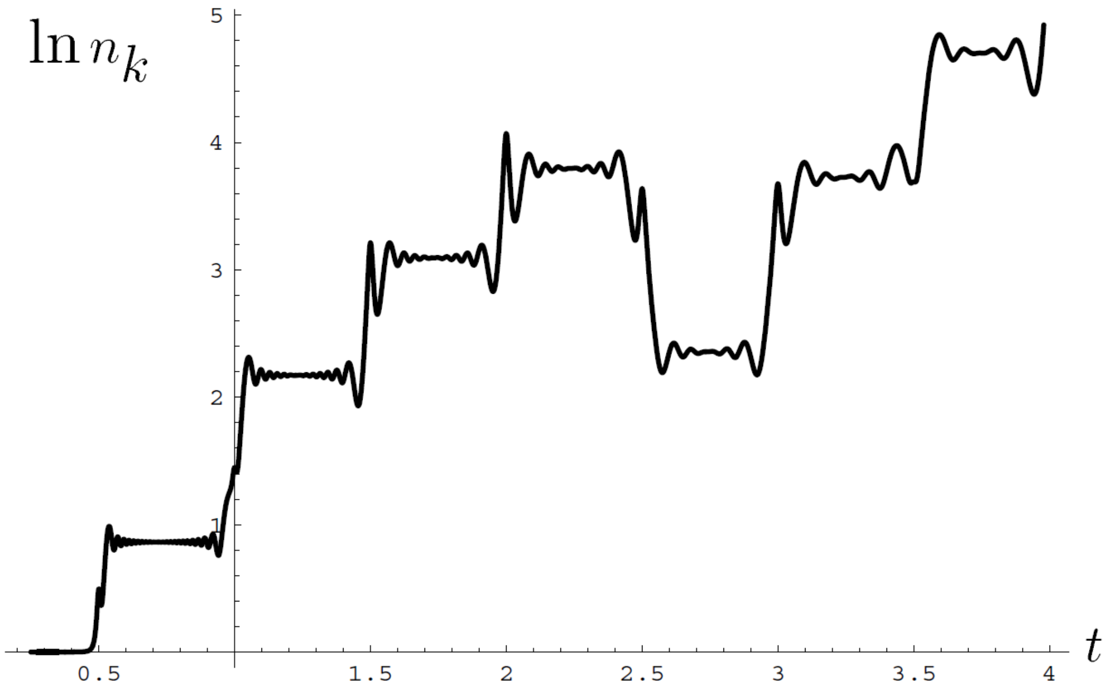

Figure III.4: Early stochastic parametric resonance in an expanding universe for the field χ as obtained by numerical methods with $a \sim t^{2/3}$, $g = 5 \cdot 10^{-4}$ and $q_0 \sim 3 \cdot 10^3$, t is depicted in units of $2\pi/m$; reproduced from [1] by *L. Kofman, A. Linde, A. Starobinsky*, 1997

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