

The conformal anomaly

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Abstract

This article shows the rise of the conformal anomaly in a 4 dimensional massless scalar quantum field theory coupled to a classical gravitational background. To this end, a careful regularization and renormalization procedure has to be performed due to the divergent behavior of the stress-energy tensor $\langle T_{\mu\nu} \rangle$.

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1 Introduction

The starting point of our analysis is a scalar field theory coupled to a gravitational background with the action [1]

$$S = S_m + S_{grav} = \int d^n x \sqrt{-g_x} \left(\frac{1}{2} (\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 - R \xi \phi^2) + \frac{R - 2\Lambda}{16\pi G} \right). \quad (1)$$

Where R is the Ricci scalar and ξ the coupling. On a classical level, the equations of motion are obtained by performing a variation of the total action with respect to the metric,

$$\frac{2}{\sqrt{-g_x}} \frac{\delta S}{\delta g^{\mu\nu}} = 0. \quad (2)$$

This results in the Einstein field equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}. \quad (3)$$

Since we want to study the conformal symmetry breaking during the transition from the classical to the quantum level, our interest is in the quantum analog of (3),

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu} \rangle. \quad (4)$$

Eventually we treat the matter content (i.e. the scalar field) quantum mechanically and the gravitational background classically. The subscript B indicates that we are dealing with the non-renormalized and therefore not measurable quantities. The concrete meaning of the bracket will be explained in a moment.

While taking the step from (3) to (4) a problem occurs. This problem is already familiar from flat space quantum field theory. The expectation value of the stress-energy tensor is a divergent quantity [1],[5]. However, in Minkowski space this is not a problem, as we can renormalize the Lagrangian by an infinite amount, i.e. we simply discard the zero point energy. This is possible as in a theory without gravity only energy differences are measurable. In a gravitational theory, we cannot do this, since according to (4) energy itself is the source of gravity, and will bring out the very spacetime curvature whose effects we are trying to study.

In order to arrive at a finite and physical meaningful result, we need to deal with the divergent behavior of $\langle T_{\mu\nu} \rangle$, i.e. we need to perform the techniques of regularization and renormalization. This aspect will be particularly considered in the second chapter. In the third chapter it will be shown that the regularization and renormalization procedure lead to a breaking of the conformal symmetry of the quantum theory.

2 Regularization and Normalization

In order to get from (3) to (4) we replace S_m with the so called *effective action* W , which gives us exactly the quantum analog of $T_{\mu\nu}$ if we perform the variation principle,

$$\frac{2}{\sqrt{-g_x}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle. \quad (5)$$

Now our task is to find an explicit form for W . To this end, we consider the generating functional [1]

$$\langle 0, out | 0, in \rangle = Z[J] = \int \mathcal{D}\phi e^{iS_m + i \int d^n x J(x) \phi(x)}, \quad (6)$$

which can be interpreted as the vacuum to vacuum persistence amplitude. Considering the variation of the sourceless functional yields

$$\delta Z[0] = \int \mathcal{D}\phi \delta S_m e^{iS_m} = i \langle 0, out | \delta S_m | 0, in \rangle_{J=0}. \quad (7)$$

Performing the derivative leaves us with

$$\frac{2}{\sqrt{-g_x}} \frac{\delta Z[0]}{\delta g^{\mu\nu}} = \langle 0, out | T_{\mu\nu} | 0, in \rangle_{J=0}. \quad (8)$$

Motivated by this result we define the *effective action* as

$$W = -i \ln(Z[0]). \quad (9)$$

Checking the variation principle with respect to (9) reveals the nature of the brackets encountered in (4),

$$\frac{2}{\sqrt{-g_x}} \frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle 0, out | T_{\mu\nu} | 0, in \rangle_{J=0}}{\langle 0, out | 0, in \rangle_{J=0}} = \langle T_{\mu\nu} \rangle. \quad (10)$$

Our next task is to evaluate (9). For this purpose we need to compute $Z[0]$ [5],

$$\begin{aligned} Z[0] &= \int \mathcal{D}\phi e^{iS_m} \\ &= \int \mathcal{D}\phi e^{\frac{-i}{2} \int d^n x \sqrt{-g_x} \phi_x (\Box_x + m^2 + R\xi - i\epsilon) \phi_x} \\ &= \int \mathcal{D}\phi e^{\frac{-i}{2} \int d^n x d^n y \sqrt{-g_x} \sqrt{-g_y} \phi_x (\Box_x + m^2 + R\xi - i\epsilon) \delta^n(x-y) \frac{1}{\sqrt{-g_y}} \phi_y} \\ &= \int \mathcal{D}\phi e^{\frac{-i}{2} \int d^n x d^n y \sqrt{-g_x} \sqrt{-g_y} \phi_x K_{xy} \phi_y}. \end{aligned}$$

Where we have integrated by parts and defined the kernel

$$K_{xy} := (\square_x + m^2 + R\xi - i\epsilon) \delta^n(x - y) \frac{1}{\sqrt{-g_y}}. \quad (11)$$

Now, we use the defining equation for the Green function

$$(\square_x + m^2 + R\xi - i\epsilon) G_F(x, x') = \frac{-1}{\sqrt{-g_x}} \delta^n(x - x'), \quad (12)$$

and the inverse operator for the kernel K_{xy} defined as

$$\int d^n y \sqrt{-g_y} K_{xy} K_{yz}^{-1} = \delta^n(x - z) \frac{q}{\sqrt{-g_z}}. \quad (13)$$

Combining (11) and (12) we get the important result

$$K_{xy}^{-1} = -G_F(x, y). \quad (14)$$

In order to evaluate our expression for $Z[0]$ we perform a coordinate transformation

$$\phi' = \int d^n y \sqrt{-g_y} \sqrt{K_{xy}} \phi_y. \quad (15)$$

With this transformation we get

$$Z[0] = A \times \det \left(\sqrt{K} \right)^{-1}. \quad (16)$$

The first term contains the transformed functional integral and the second term is the Jacobian. At this point, we notice that in the end we are interested in the derivative of $\ln Z[0]$ with respect to the metric. Since the first term in (16) is independent of the metric we can simply discard it and we arrive at

$$W = -i \ln \left(\det \left(\sqrt{K} \right)^{-1} \right). \quad (17)$$

With the help of (14) and after some algebra we can rewrite (17) and get

$$W = \frac{-i}{2} \text{tr} (\ln (-G_F)). \quad (18)$$

At this point, it is important to notice that G_F in (18) is to be interpreted as an operator which acts on a space of states defined as

$$G_F(x, x') = \langle x | G_F | x' \rangle. \quad (19)$$

The trace appearing in (18) can be computed via

$$\text{tr} M = \int d^n x (\sqrt{-g_x} \langle x | M | x \rangle). \quad (20)$$

In order to get a result for (18) we need an explicit representation for G_F . We will use the so-called DeWitt-Schwinger representation of G_F [1],[3]

$$G_F^{DS}(x, x') = -i\Delta^{1/2}(x, x')(4\pi)^{-n/2} \int_0^\infty ds i(is)^{-n/2} e^{-im^2 s + \frac{\sigma}{2is}} F(x, x'; is). \quad (21)$$

Where $\Delta^{1/2}(x, x')$ is the so-called Van Vleck determinant, which will vanish in the UV-limit that we will consider in the following [1]. σ is one-half of the square of the proper distance between x and x' .

The last factor represents the adiabatic expansion in terms of the proper time s

$$F(x, x'; is) = a_0(x, x') + (is)a_1(x, x') + (is)^2 a_2(x, x') + \dots \quad (22)$$

The explicit form of the DeWitt-coefficients a_i will be given later.

For evaluating (18) we need two more ingredients. First we rewrite the operator analog of (14) as an integral equation

$$G_F = -K^{-1} = -i \int_0^\infty ds e^{-iKs}. \quad (23)$$

If we apply two different states on (23) we can identify the integrand of the right-hand side with the integrand of (21)

$$\langle x | e^{-iKs} | x' \rangle = i(4\pi)^{-n/2} \Delta^{1/2} e^{-im^2 s + \frac{\sigma}{2is}} F(x, x'; is) (is)^{-n/2}. \quad (24)$$

Now let us consider the following integral (i.e. exponential integral function)

$$\int_\Lambda^\infty ds e^{-iKs} \frac{1}{s} = -Ei(-i\Lambda K) \approx -\ln(K) - \gamma - \ln(i\Lambda). \quad (25)$$

Where γ is Euler's constant. If we take the limit $\Lambda \rightarrow 0$, the third term will diverge, but since we still are only interested in derivatives with respect to the metric we can forget about this infinite constant. After taking the limit we arrive at

$$\int_0^\infty ds e^{-iKs} \frac{1}{s} = -\ln(K) = \ln(-G_F). \quad (26)$$

Now we can combine (24) with (26) and get

$$\begin{aligned} \langle x | \ln(-G_F) | x' \rangle &= \int_0^\infty i/s (4\pi)^{-n/2} \Delta^{1/2} e^{-im^2 s + \frac{\sigma}{2is}} F(x, x'; is) (is)^{-n/2} \\ &= - \int_{m^2}^\infty dm^2 G_F^{DS}(x, x'). \end{aligned}$$

Where the integral over m^2 gives us the additional $1/s$.
Combining this result with (18) and (20) provides our master formula for the *effective action*

$$W = i/2 \int d^n x \sqrt{-g_x} \lim_{x \rightarrow x'} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x'). \quad (27)$$

From (27) we can immediately read off the Lagrangian

$$L_{eff} = i/2 \lim_{x \rightarrow x'} \int_{m^2}^{\infty} dm^2 G_F^{DS}(x, x'). \quad (28)$$

Inspecting (28) reveals that the Lagrangian diverges at the lower end of the s integral because the $\sigma/2s$ damping factor vanishes in the limit $x \rightarrow x'$. The divergences in L_{eff} are, of course, the same that afflict $\langle T_{\mu\nu} \rangle$. At this point, we can try to isolate the divergent terms in L_{eff} and perform the renormalization.

Let us rewrite (28) explicitly,

$$L_{eff} = \lim_{x \rightarrow x'} \frac{\Delta^{1/2}(x, x')}{2(4\pi)^{n/2}} \sum_{j=0}^{\infty} a_j(x, x') \int_0^{\infty} ds i(is)^{j-1-n/2} e^{-i(m^2 s - \sigma/(2s))}. \quad (29)$$

For L_{eff} to be divergent, two things have to happen. We have to take the coincident limit $x \rightarrow x'$ and we have to specify the dimensionality n . If we take the physical limit $n \rightarrow 4$, the first three terms will be divergent.

Our strategy will be to keep the dimensionality arbitrary and take the coincident limit, i.e. we perform dimensional regularization. If we do so, we can evaluate the integral explicitly and get

$$L_{eff} = 1/2(4\pi)^{-n/2} \sum_{j=0}^{\infty} a_j(x) (m^2)^{n/2-j} \Gamma(j - n/2). \quad (30)$$

Where Γ is the Gamma-function. As usual in dimensional regularization, we introduce an additional mass scale μ in order to retain the units of L_{eff} as $(\text{length})^{-4}$, even when $n \neq 4$, which provides us with

$$L_{eff} = 1/2(4\pi)^{-n/2} (m/\mu)^{n-4} \sum_{j=0}^{\infty} a_j(x) m^{4-2j} \Gamma(j - n/2). \quad (31)$$

As $n \rightarrow 4$, the first three terms of (31) diverge because of poles in the Γ -functions. Calling these three terms L_{div} , we obtain

$$L_{div} = -(4\pi)^{n-4} \left(\frac{1}{n-4} + \frac{1}{2} (\gamma + \ln(m^2/\mu^2)) \right) \left(\frac{4m^4 a_0}{n(n-2)} - \frac{2m^2 a_1}{n-2} + a_2 \right). \quad (32)$$

Where we expanded the Γ -functions and used the expansion

$$(m/\mu)^{n-4} = 1 + 1/2(n-4)\ln(m^2/\mu^2) + O((n-4)^2), \quad (33)$$

and dropped terms that vanish when $n \rightarrow 4$. The DeWitt-coefficients in the coincident limit are given by [1]

$$a_0 = 1, \quad (34)$$

$$a_1 = \left(\frac{1}{6} - \xi\right)R, \quad (35)$$

$$a_2 = \frac{1}{180}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{1}{180}R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{6}\left(\frac{1}{5} - \xi\right)\square R + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2R^2. \quad (36)$$

Inspecting (34),(35) and (36) makes it obvious that L_{div} is a purely geometrical expression. Motivated by this observation, we absorb the divergent part of the Lagrangian (32) in the gravitational part of the total action:

$$S = S_{grav} + W = (S_{grav} + W_{div}) + (W - W_{div}) = S_{g,ren} + W_{ren} \quad (37)$$

More precisely, we add the divergent term in (32) to the different bare constants in the original Lagrangian and define the resulting constants as the renormalized quantities. The new action (37) leads to the renormalized and finite result of our semi-classical theory [1],[3]:

$$G_{\mu\nu} + \Lambda_R + \alpha_R H_{\mu\nu}^{(1)} + \beta_R H_{\mu\nu}^{(2)} = -8\pi G_R \langle T_{\mu\nu}^{ren} \rangle \quad (38)$$

Interestingly, the coupling of the gravitational field with a quantum field seems to introduce higher order corrections to the Einstein theory, i.e. the third and fourth terms on the left-hand side of (38), which correspond to the term in L_{div} proportional to a_2 . Since these terms are not present in the original action the right way to deal with them is the following: We introduce two additional bare couplings α_B and β_B into the original action and absorb the divergent terms in them. The resulting renormalized quantities α_R and β_R are the measurable higher order corrections.

3 The conformal anomaly

Now, we will investigate the conformal anomaly. In general, only a massless theory can be conformal invariant, since any mass scale would act as a reference point. Since we are interested in the physical limit $n = 4$ we will modify our previous theory in the following way:

$$m = 0 \quad (39)$$

$$\xi = \frac{1}{6} \quad (40)$$

Such a theory is invariant under a conformal transformation

$$g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = \Omega^2(x)g_{\mu\nu}(x). \quad (41)$$

This is basically a rescaling of the metric. The analog infinitesimal transformation is given by

$$\delta g_{\mu\nu} = 2g_{\mu\nu}\delta\Omega. \quad (42)$$

If we assume a conformal invariant action then its variation with respect to an infinitesimal transformation has to vanish [4], i.e.

$$0 = \delta S = \int d^n x \frac{\delta S}{\delta g^{\mu\nu}} \delta g^{\mu\nu} = \int d^4 x \sqrt{-g_x} T^{\mu\nu} g_{\mu\nu} \delta\Omega. \quad (43)$$

As (41) has to be true for all variations $\delta\Omega$ the integrand has to vanish, i.e.

$$T^\mu_\mu = 0. \quad (44)$$

Means that if the theory is conformal invariant then the stress-energy tensor is traceless. We will rely heavily on this result. Starting with the conditions (39),(40) and (44) we get

$$\frac{2}{\sqrt{-g_x}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T^\mu_\mu \rangle = 0 \quad (45)$$

for the total action. Let us go back to equation (31) and consider the massless and $n = 4$ limit. If we do so, we see that the first two terms vanish. The third term is the only UV-divergent term, since it is the only term having a negative exponent in the $n = 4$ limit. Therefore, we can write the divergent part of the action in the following way:

$$W_{div} = \frac{1}{2}(4\pi)^{-n/2}(m/\mu)^{n-4}\Gamma(2-n/2) \int d^4 x \sqrt{-g_x} a_2(x) \quad (46)$$

$$= \frac{1}{2}(4\pi)^{-n/2}(m/\mu)^{n-4}\Gamma(2-n/2) \int d^4 x \sqrt{-g_x} (1/120 F(x) - 1/360 G(x)) \quad (47)$$

Where we have defined

$$F(x) = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta}R^{\alpha\beta} + 1/3R^2 \quad (48)$$

$$G(x) = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2 \quad (49)$$

To obtain (47) we have dropped the terms proportional to $\square R$ and R^2 in $a_2(x)$, because the first term is a total divergence and hence will not contribute to the action, the second as its coefficient is proportional to $(n-4)^2$.

The important thing to notice is that in $n=4$ (and only in 4) F is the square of the Weyl tensor $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ [1],[3]. Moreover $\int d^4x \sqrt{-g_x} G$ is according to the Gauss-Bonnet Theorem equal to the Euler characteristic [1],[3]. That means that both of these quantities remain invariant under conformal transformations. It follows that in 4 dimensions W_{div} is conformal invariant.

However, we must not relax the regularization and pass to $n=4$ before computing the physical quantities even when we put $n=4$ at the end of the calculation. That is what we are doing now. We compute the quantum stress-energy tensor away from the physical dimension and take the limit after the computation. This is the operational origin of the conformal anomaly. The result is

$$\langle T_\mu^\mu \rangle = \frac{2}{\sqrt{-g_x}} g^{\mu\nu} \frac{\delta W}{\delta g^{\mu\nu}} \quad (50)$$

$$= \frac{1}{16\pi^2} [1/120(F - 2/3\square R) - 1/360G] \quad (51)$$

$$= \frac{a_2}{16\pi^2}. \quad (52)$$

Since we started with a conformal invariant action W and therefore traceless stress-energy tensor

$$0 = \langle T_\mu^\mu \rangle = \langle T_\mu^\mu \rangle_{div} + \langle T_\mu^\mu \rangle_{ren}, \quad (53)$$

we immediately see

$$\langle T_\mu^\mu \rangle_{ren} = -\frac{a_2}{16\pi^2}. \quad (54)$$

This is the celebrated trace anomaly. Although we started with a conformal invariant action on the classical level, the symmetry on the quantum level breaks.

From our previous analysis, it is not difficult to achieve a result valid for arbitrary dimensionality [1]

$$\langle T_\mu^\mu \rangle_{ren} = -\frac{a_{n/2}}{(4\pi)^{n/2}}. \quad (55)$$

The underlying reason for the symmetry breaking is the introduction of the auxiliary mass scale during the regularization procedure

[1],[3].

Another way to understand the anomaly is the path integral formulation of QFT. The quantum theory would be conformally invariant if a path integral were invariant under conformal transformations. However, this is impossible because one cannot choose the integration measure to be conformally invariant and at the same time generally covariant[4].

4 References

References

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