Hawking Radiation

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In this report the aim is to understand the behaviour of a Schwarzshild Black Hole in a context where quantum field fluctuations are taken into account. We are going to see how to introduce briefly the idea of Black Holes, how extraction of energy from them is possible and the Schwarzschild solution for the metric seen by a distant observer. Then we are going to change coordinates to a near horizon observer to demonstrate that that point is not a singularity, and draw the Kruscal Diagram.
In the second part we consider scalar field fluctuations around the horizon as seen by the two observers, define two different vacuums that lead a non-zero number particle density. This means that the Black Hole emits radiation, and in the empty space ultimately evaporates.
In the last part we give some numbers of lifetime of Black Holes and introduce the Black Hole Thermodynamics.

Chapter 1

Black Holes

The first intuitive idea of a Black Hole is a highly massive object with a strong gravitational field that even light light cannot escape from. There are several solutions for Black Holes, depending on the characteristics they have (momentum, charge). The easiest one is the non-charged, non-rotating solution, represented by the Schwartschild metric:

$$ds^2 = \left(1 - \frac{2GM/c^2}{r}\right)dt^2 - \left(1 - \frac{2GM/c^2}{r}\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{1.1}$$

This is the metric measured by a far distant observer in polar coordinates in respect to the Black Hole center $r = 0$, with M its mass and $d\Omega$ the angular coordinates. Here we see that $r_g \equiv 2GM/c^2$ is a particular radius called *event horizon*, where the metric diverges and it represents a no-turn-around point. Once an object crosses this distance, it's doomed not to come out again, and fall inside the central point at $r = 0$, the other singularity.
According to General Relativity a Black Hole can only absorb matter (or equivalently radiation), thus its mass can only increase. This solution is called eternal because the object will never disappear.
Let us now consider Quantum Mechanics. It is known that vacuum isn't exactly empty, each instant virtual couples of particle-antiparticle are created by vacuum fluctuations and annichiliate between themselves.
In 1974 Stephen Hawking linked these concepts. He considered quantum fields in a Black Hole background, and he discovered that a Black Hole emits thermal particles and in this way it can *evaporate*. Why is this concept new (and suprising)?

1.1 Rotating Black Holes

In principle the idea of a Black Hole emitting energy is not strange. If we consider a Rotating Black Hole, described by the Kerr metric, we can see that there

are *negative energy regions* outside the event horizon, so that the gravitational field can convert a *virtual* couple particle-antiparticle into a *real* pair. In this way, because of the total energy of the particle is to be zero, one particle (with negative energy) falls inside the black hole while the other (with positive energy) can escape to infinity.
Let us explain further about how it is possible for the pair to gain energy from Kerr's Black Hole.

- In Classical Mechanics a rotating body with a medium (for example air) exitates the particles around, transferring to them energy thus losing it. In this way the medium acquires energy and the body ultimately stops its rotation. If we are in vacuum, no energy loss is possible.

- In Quantum Mechanics a rotating body in vacuum can amplify the fluctuations and they gaining energy from the body become a real radiation that an observer around the body can see.

In this way the gravitational tidal forces around the Black Hole pull apart the virtual pairs (that would quickly annichiliate if not for these forces) and create a population of real particles.

1.2 Non-Rotating Black Holes

We have seen that a radiation coming from a rotating Black Hole is reasonable. Problems arise when we consider no angular momentum, because no negative-energy regions exist outside the event horizon. Nevertheless we are going to see why a production of particles is still possible, and a first pictorial explanation can be given immediately. We can see that negative energy regions are *inside* the horizon, so a couple created in the proximity of it has this situation:

- Negative energy particle is created *inside* the Black Hole, it falls inside

- Positive energy particle is created *outside* the Black Hole, it escapes to infinity

In this way the Black Hole can emit radiation, and its mass evaporates.
We can give a rough estimate of the outgoing radiation: its energy (from uncertainity principle) $E \leq \hbar c / r_g$ lead us to see that the De Broglie wavelengh is bigger than the event horizon, so the probability to escape is bigger than zero. The *temperature* is $T_H \sim \hbar c / r_g k_B = \hbar c^3 / G M k_B$. We point out that this is a very naive guess of T_H! We need to go more in detail through the Schwarzshild solution and the quantum fields to have a better view of the Hawking radiation.

Chapter 2

Schwarzschild solution

In this chapter we are going to study the metric of a Schwarzschild Black Hole. We introduce natural units: $G = \hbar = c = 1$ and we restrict the (1.1) to 2-dimensional case, considering just the time and radius-coordinates. This is because the 2 addictional angular dimensions don't add physical information for the process we are interested. So the $2 - d$ Schwarzshild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) dt^2 - \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 \tag{2.1}$$

with $r_g = 2M$. As we have seen before, this metric is singular in:

- $r = r_g$, event horizon. This is not a physical singularity, since one can compute the scalar curvature in that point and see that it is finite. The metric is well defined just for $r > r_g$.
- $r = 0$, true singularity.

We want to show that $r = r_g$ is just a coordinate singularity, so we have just to change them.

2.1 Tortoise coordinates

We would like to have

$$dr^* = \left(1 - \frac{r_g}{r}\right)^{-1} dr \tag{2.2}$$

so we define the Tortoise coordinates:

$$r(r^*) = r - r_g + r_g * ln\left(r/r_g - 1\right) \tag{2.3}$$

These are still defined for $r > r_g$, but now $-\infty < r^* < +\infty$ with

- $r \to r_g \Rightarrow r^* \to -\infty$

- $r \to +\infty \Rightarrow r^* \to +\infty$

The metric now is:

$$ds^2 = \left(1 - \frac{r_g}{r(r^*)}\right)(dt^2 - dr^{*2}) \tag{2.4}$$

where $r(r^*)$ comes from (2.3). Introducing now the lightcone coordinates:

- $\tilde{u} = t - r^*$
- $\tilde{v} = t + r^*$

the metric becomes:

$$ds^2 = \left(1 - \frac{r_g}{r(\tilde{u}, \tilde{v})}\right) d\tilde{u} d\tilde{v} \tag{2.5}$$

At this point we still have a singularity at $r = r_g$, and we can cover just the exterior of the Black Hole. We need one more change of coordinates to get rid of it.

2.2 Kruscal-Szekeres coordinates

Using (2.3) and the new lightcone coordinates we find an useful relation:

$$\left(1 - \frac{r_g}{r}\right) = \frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{\tilde{v} - \tilde{u}}{2r_g}\right)} \tag{2.6}$$

We then describe the metric as:

$$ds^2 = \frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{\tilde{v}}{2r_g}\right)} e^{\left(\frac{-\tilde{u}}{2r_g}\right)} d\tilde{u} d\tilde{v} \tag{2.7}$$

We define then the Kruscal-Szekeres lightcone coordinates, of a free falling object near the horizon:

- $u = -2r_g * e^{\left(\frac{-\tilde{u}}{2r_g}\right)}$
- $v = 2r_g * e^{\left(\frac{\tilde{v}}{2r_g}\right)}$

and the metric:

$$ds^2 = \frac{r_g}{r(u, v)} e^{\left(1 - \frac{r(u,v)}{r_g}\right)} du dv \tag{2.8}$$

We can see now that it is regular in $r = r_g$, so the event horizon was just a coordinate problem, and a free falling observer doesn't feel anything particular crossing it. We can see that these coordinates are still defined for $r > r_g$ but now can be naturally extended everywhere:

- $-\infty < v < 0 \Rightarrow -\infty < v < +\infty$
- $0 < u < +\infty \Rightarrow -\infty < u < +\infty$

Now using (2.3) and (2.6) we find the relation between the old far away observer coordinates (t, r) and the free falling ones (u, v):

$$uv = -4r_g^2 e^{\frac{r^*}{r_g}} = -4r_g^2 \left(\frac{r}{r_g} - 1 \right) e^{\frac{r}{r_g} - 1} \tag{2.9}$$

$$(\frac{v}{u})^2 = e^{\frac{2t}{r_g}} \tag{2.10}$$

Here the (2.9) tells us that at the horizon $r = r_g \Rightarrow u = 0$ <u>or</u> $v = 0$, so we have <u>two</u> horizons.
The (3.10) tells us that:

- $v = 0 \Rightarrow t = -\infty$ <u>past</u> horizon
- $u = 0 \Rightarrow t = +\infty$ <u>future</u> horizon

We can introduce timelike and spacelike coordinates:

- $u = T - R$
- $v = T + R$

Now it can be useful to see a grafical representation of the spacetime we are describing, a Schwarzschild Black Hole can be portaited by the Kruscal Diagram.

We can see that the two horizons divide the spacetime in 4 different regions, in region I the trajectories with constant distance from the central singularity are timelike, while in region II they are spacelike, and we can see the $r = 0$ singularities in the past and in the future.
A quick remark: the Black Holes we find in nature are created during the collapse of a star, so no past singularity should exist. Therefore just the I-II regions should be physically relevant, and the III-IV can be ignored in this case.

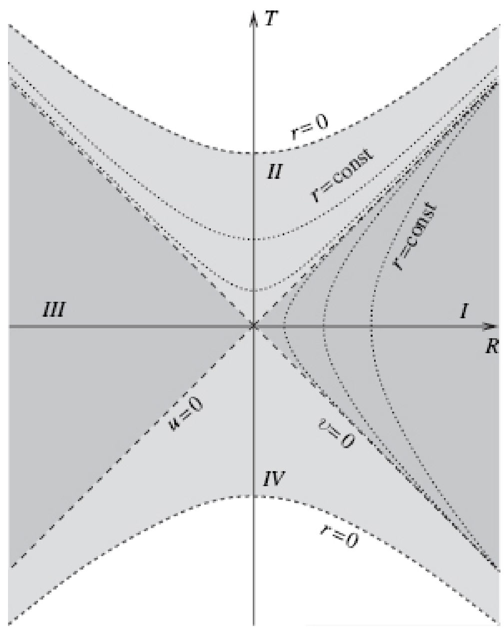

Figure 2.1: Shaded regions IIV are the different asymptotic domains of the spacetime. Dashed lines represent the horizon u = 0 and v = 0. Dotted lines are surfaces of constant r. Thick dotted lines represent the singularity r = 0.

Chapter 3

Field Quantization and Hawking radiation

We have seen how the Schwarzshild Black Hole can be described by different observers in different coordinates, and how the event horizon is completely different for the two. Now we can start to add to this background the quantum fields fluctuation, to see what these two observe near the Black Hole horizon.
We consider for simplicity a free scalar field, but this description can be repeated for other particles, like fermions.
The action:

$$S[\phi] = \frac{1}{2}\int g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\sqrt{-g} * d^2x \tag{3.1}$$

This is conformally invariant, which means that it doesn't change if we change the coordinates in which we express the field. Solving the equation of motion we find the solution for the field with the two sets of coordinates $(\tilde{u}, \tilde{v})$ and (u, v):

- $\phi = \tilde{A}(\tilde{u}) + \tilde{B(\tilde{v})}$
- $\phi = A(u) + B(v)$

with $\tilde{A}$, $\tilde{B}$, A , B arbitrary smooth functions. We can express the field to respect to the modes (Fourier transformation) as:

- $\phi \propto e^{-i\Omega\tilde{u}} = e^{-i\Omega(t-r^*)}$
- $\phi \propto e^{-i\omega u} = e^{-i\omega(T-R)}$

It's now the point we should ask ourselves what a particle is: positive frequency modes with respect to the proper time of the observer. So:

- A distant observer measures a $t \Rightarrow$ he sees a virtual radiation

- A near-horizon observer measures a $T \Rightarrow$ he sees a real radiation!

We are going to see that this last observes a thermal bath with finite temperature that pop out of the local acceleration horizon, turn around and free-fall back in.
Let's distinguish now.

3.1 Near-horizon observer

For $r \to r_g$ a near-horizon observer measures a metric:

$$ds^2 \to dudv = dT^2 - dR^2 \tag{3.2}$$

so as said he measures a time T. The field operator expansion:

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u} \hat{a}_\omega^- + e^{+i\omega u} \hat{a}_\omega^+ \right] + [v - modes] \tag{3.3}$$

where we don't write down the incoming $v - modes$ for simplicity.
We then define a vacuum $|0_K>$ that is annichilated by:

$$\hat{a}_\omega^- |0_K >= 0 \tag{3.4}$$

called *Kruscal vacuum*. This state contains particles for the distant observer (we are going to see the number density). The vacuum is well-defined, because there is no singularity at the horizon $r = r_g$, so we can think this as a physical vacuum.

3.2 Distant observer

For $r \to \infty$ a distant observer measures a metric:

$$ds^2 \to d\tilde{u}d\tilde{v} = dt^2 - dr^{*2} \tag{3.5}$$

so as said he measures a time t. The field operator expansion:

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{\sqrt{2\pi}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega\tilde{u}} \hat{b}_\Omega^- + e^{+i\Omega\tilde{u}} \hat{b}_\Omega^+ \right] + [v - modes] \tag{3.6}$$

where again we don't write down the incoming $v - modes$ for simplicity.
We then define a vacuum $|0_B>$ that is annichilated by:

$$\hat{b}_\Omega^- |0_B >= 0 \tag{3.7}$$

called *Boulware vacuum*. This state contains no particles for this observer (the number density is 0). The problem is that this vacuum is not physical.
In order to understand why, we should remember that this set of coordinates

is singular in the horizon, so at this point the expectation value of the energy density diverges. From:

$$< 0_K|(\partial_u \hat{\phi})^2|0_K >=< 0_B|(\partial_{\tilde{u}} \hat{\phi})^2|0_B > \quad (3.8)$$

because of the conformal invariance of the action, then:

$$< 0_B|(\partial_u \hat{\phi})^2|0_B >= (\frac{\partial \tilde{u}}{\partial u})^2 < 0_B|(\partial_{\tilde{u}} \hat{\phi})^2|0_B > \quad (3.9)$$

We have calculated the expectation value of the energy density for a near-horizon observer in this vacuum, but the Jacobian $(\frac{\partial \tilde{u}}{\partial u})^2$ diverges at $u = 0$, so the energy density.

3.3 Hawking radiation

We have seen the difference between the vacuums of the two observers. Now that we have seen that our true physical vacuum is Kruscal's, we can ask ourselves the *number* of particles in this vacuum for a distant observer:

$$< \hat{N}_\Omega >=< 0_K|\hat{b}^+_\Omega \hat{b}^-_\Omega|0_B >= \left[e^{\frac{2\pi\Omega}{k}} - 1\right]^{-1} \delta(0) \quad (3.10)$$

with $k = 1/2r_g$ called *surface gravity* and the $\delta(0)$ simply representing the volume of the space, so the number density is:

$$< \hat{n}_\Omega >= \left[e^{\frac{2\pi\Omega}{k}} - 1\right]^{-1} \quad (3.11)$$

which is the Planckian distribution, so we can see that the radiation is thermal with temperature that depends just on the Black Hole mass:

$$T_H = \frac{k}{2\pi} = \frac{1}{8\pi M} \quad (3.12)$$

We have seen that $|0_K >$ contains the right-moving, so outgoing particles from the Black Hole, but also left-moving, so ingoing ones, with the same thermal spectrum.
The fact that Black Holes has a population of real particles popping out and going back inside the horizon means that *quantum fields in a Black Hole background are consistent only if the body is in a thermal reservoir with a temperature* T_H.
Then since a Black Hole absorbs particles has to emit to mantain equilibrium so one in empty space must evaporate emitting thermal radiation.

3.4 Non-eternal Black Holes

A slightly different approach is to be made if we consider Black Holes as final result of a stellar collapse. In fact, in this case the $v = 0$ horizon doesn't exist, so there is no physical need of choosing for the v-modes the Kruscal vacuum. To understand why, in the past (before the collapse) the spacetime was almost flat, so the Boulware vacuum wasn't singular because no horizon existed. Then a consistent way of choosing the vacuums should be:

- right-moving modes: $\hat{a}^-_\omega$ that annichilates $|0_K>$
- left-moving modes: $\hat{b}^-_\Omega$ that annichilates $|0_B>$

3.5 3+1-dimensions

We can take a brief look on what happens if we consider again 3+1-dimensions. The description of the problem is identical at the one just seen, we can write down the action and decompose the scalar field in spherical harmonics $\phi(t, r, \theta, \varphi) = \sum_{lm} \phi_{lm}(t, r) Y_{lm}(\theta, \varphi)$ and substitute in the equation of motion $\Box\phi = 0$. We find:

$$\left[{}^{(2)}\Box + \left(1 - \frac{r_g}{r}\right) \left(\frac{r_g}{r^3} + \frac{l(l+1)}{r^2}\right)\right] \phi_{lm}(\theta, \varphi) = 0 \tag{3.13}$$

where we recognize the contribute of the 1+1-dimension case plus a correction term $V_l(r)$ that is not 0 even for $l = 0$. This acts as a potential that the outgoing wave has to surpass in order to arrive to infinity. In general this has the consequence of lowering the number density of particles seen by an observer:

$$< \hat{n}_\Omega >= \Gamma_l(\Omega) \left[e^{\frac{\Omega}{T_H}} - 1\right]^{-1} \tag{3.14}$$

where $\Gamma_l(\Omega)$ is a greybody factor due to $V_l(r)$, always < 1, that depends on the field considered.
The physics of the Hawking radiation can then be seen simply by the 1+1-dimension case, because $V_l(r)$ acts just outside the Black Hole so it's no direct consequence of the process.

Chapter 4

Thermodynamics of Black Holes

In light of the new understanding of the thermal interactions between a Black Hole and its surroundings, we can briefly try to understand how this radiation excange works.

4.1 Mass loss

When we consider a Black Hole in empty space it emits radiation and loses mass. But how?
The flux of radiated energy from a spherical Black Hole with surface area $\Sigma = 4\pi r_g^2 = 16\pi M$ and temperature $T_H = 1/8\pi M$ is given by the Stefan-Boltzmann law:

$$L = \Gamma\gamma\sigma T_H^4 \Sigma = \frac{\Gamma\gamma}{15360\pi M^2} \tag{4.1}$$

with Γ corrects for the greybody factors, γ is the number of massless degrees of freedom and σ is the Stefan-Boltzmann constant in Planck units.
Then the mass decreases as:

$$\frac{dM}{dt} = -L = -\frac{\Gamma\gamma}{15360\pi M^2} \Rightarrow M(t) = M_0\sqrt{1 - t/t_L} \tag{4.2}$$

with $t_L = 5120\pi M_0^3/\Gamma\gamma$. We see then that the lifetime of a Black Hole is $t_L \propto M_0^3$ so the mass to the third. So since its mass it's enourmous, it takes a long time to evaporate, and while this happens the temperature increases.
To give some numbers, let's imagine a $M_0 = M_\odot$, then $t_L \sim 10^{67}$ years which is far bigger than the age of the universe $\sim 10^{10}$ years. This means that at our current age it's not possible to observe the evaporation of such massive objects, then we should look for $M_0 \sim 10^{15} g$ Black Holes to see something

today. The problem is that these objects are not produced by currently known stellar collapse.
It is important to point out that the evaporation process can change drastically because when we reach $M \sim 10^{-5}$ the scale of the Black Hole is the Planck Lenght, and a Quantum Gravity Theory is necessary to describe what happens. It may be possible an analogy with the Hydrogen atom, where at the lowest orbit the electron doesn't radiate anymore, so the Black Hole stop radiating and doesn't disappear.

4.2 Entropy

Let's now introduce briefly the concept of Black Hole intrinsic entropy S_{BH}. Let's say the Black Hole has $S_{BH} = 0$, if it assorbs matter with $S \neq 0$, then the total entropy of the universe is diminished, but this is not possible. The solution is $S_{BH} \neq 0$, linked to the surface area $\Sigma = 16\pi M^2$. Differentiating:

$$dM = \frac{1}{8\pi M} d(\frac{\Sigma}{4}) \tag{4.3}$$

which has the same form of the *I law of thermodynamics*:

$$dE = TdS \tag{4.4}$$

with $dM = dE$, $T_H = 1/8\pi M$ so we find $S_{BH} = 1/4\Sigma = 4\pi M^2$. We can set now the *I law of Black Holes' thermodynamics*:

$$dM = T_H dS_{BH} \tag{4.5}$$

With a $M = M_{\odot}$ we have a $S_{\odot} \sim 10^{36}$ so a huge number of microstates hidden behind the horizon.
We can see a deep contrast between a classic and semiclassic theory here, since General Relativity shows how a Schwarzschild Black Hole is completely characterized by its mass, so should have no entropy, while here it's enormous. This tells us that more has to be understood around Black Hole's microstates.
Taking into account the intrinsic entropy we can express the *II law of Black Holes' thermodynamics*:

$$\delta S = \delta S_{matter} + \delta S_{BH} \geq 0 \tag{4.6}$$

so that the total entropy of the black hole and the matter never decreases.