

Ambiguity of the Vacuum

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REVIEW

Time Dependent Mass

As an introduction, we briefly review the quantization of the scalar field in a non-Minkowskian metric. In particular, we consider the flat FLRW metric used in general relativity for gravitational considerations, whose line element is

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) d\mathbf{x}^2$$

where $a(t)$ is the cosmological scale factor. Defining the conformal time to be

$$\eta(t) = \int_{t_0}^t \frac{dt}{a(t)} \rightarrow dt = a(\eta) d\eta$$

we are then able to rewrite our line element as

$$ds^2 = a(\eta) (d\eta^2 - d\mathbf{x}^2)$$

Thus, we are left with a Minkowski metric with an overall scale factor. Mathematically, this manifests itself as

$$g_{\mu\nu} = a^2 \eta_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{a^2} \eta^{\mu\nu}, \quad \sqrt{-g} = a^4$$

If we let $\chi \equiv a\phi$ and look at the derivatives of χ we find

$$\chi' = a'\phi + a\phi', \quad \chi'' = a''\phi + 2a'\phi' + a\phi'', \quad \Delta\chi = a\Delta\phi$$

If we multiply the above equation by a , we get

$$\begin{aligned} 0 &= (a\phi'' + 2a'\phi') - a\Delta\phi + m^2 a^2 (a\phi) \\ &= \chi'' - a''\phi - \Delta\chi + m^2 a^2 \chi \\ &= \chi'' - \frac{a''}{a} \chi - \Delta\chi + m^2 a^2 \chi \\ &= \chi'' - \Delta\chi + \left(m^2 a^2 - \frac{a''}{a} \right) \chi \\ &= (\square + m_{\text{eff}}^2) \chi \end{aligned}$$

Henceforth, we recognize that $m_{\text{eff}} = m_{\text{eff}}(\eta)$ is effectively a time dependent mass, a consequence of considering quantum field theory (QFT) in the presence of gravity, but (unless specifically stated) are free to forget the specific form it takes.

We see that this equation is analogous to the Klein-Gordon equation with a time dependent mass[4], and can write the action

$$\begin{aligned} S &= \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \\ &= \int \frac{1}{2} d^3\mathbf{x} \left\{ (\chi')^2 - (\nabla\chi)^2 - m_{\text{eff}}^2 \chi^2 \right\} \end{aligned}$$

From our PDE for χ , we can use a Fourier transform to analyze it in momentum space, where the spacial derivative becomes a constant and

$$\chi_{\mathbf{k}}'' + \omega_{\mathbf{k}}^2 \chi_{\mathbf{k}} = 0.$$

If we assume we have two solutions to this equation, x_1 and x_2 , then $v = x_1 + ix_2$ and v^* are linearly independent and form a basis of solutions in complex space. v is called a mode function, and we have a normalization condition

$$\Im\{\dot{v}v^*\} = 1$$

The solutions $\chi_{\mathbf{k}}$ are sought in a similar way to the time independent case, as a linear combination of creation and annihilation operators, now with mode functions.

Bogolyubov Transformations and the Vacuum State

Suppose, now, that we have two mode functions which solve the ODE, v and u . We can write each as a linear combination of the other,

$$v_{\mathbf{k}}^*(\eta) = \alpha_{\mathbf{k}} u_{\mathbf{k}}^*(\eta) + \beta_{\mathbf{k}} u_{\mathbf{k}}(\eta) \rightarrow |\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$$

We can thus write the solution $\chi_{\mathbf{k}}$ in terms of these new mode functions multiplied by new annihilation and creation operators, and thus these operators are related by

$$b_{\mathbf{k}}^+ = \alpha_{\mathbf{k}} a_{\mathbf{k}}^- + \beta_{\mathbf{k}}^* a_{-\mathbf{k}}^+, \quad b_{\mathbf{k}}^- = \alpha_{\mathbf{k}}^* a_{\mathbf{k}}^+ + \beta_{\mathbf{k}} a_{-\mathbf{k}}^-$$

This is called a Bogolyubov transformation.

The last thing we cover in the review is the vacuum state. We have the b -particle operator,

$$N_{\mathbf{k}}^{(b)} = b_{\mathbf{k}}^+ b_{\mathbf{k}}^-$$

such that

$$\langle 0_b | N_{\mathbf{k}}^{(b)} | 0_b \rangle = 0$$

since the constituent operators annihilate the vacuum of the associated particle. If, however, we look for the expectation value of this operator in the a -particle vacuum, we find

$$\begin{aligned} \langle 0_a | N_{\mathbf{k}}^{(b)} | 0_a \rangle &= \langle 0_a | b_{\mathbf{k}}^+ b_{\mathbf{k}}^- | 0_a \rangle \\ &= \langle 0_a | (\beta_k a_{-\mathbf{k}}^-) (\beta_k^* a_{-\mathbf{k}}^+) \\ &= |\beta_k|^2 \delta(0) \end{aligned}$$

We divide out the delta distribution under the guise of seeking a number *density*, where the infinite factor is a result of our infinite volume and we find that this is nonzero – the a -particle vacuum has a non-zero b -particle density! This fact concludes the review, and we are now free to dive into the topic with more detail.

AMBIGUITY OF THE VACUUM STATE

In Search of a Hamiltonian

The question at hand is thus: *if each vacuum state has a non-zero particle density for any other particle, which is the 'true' physical vacuum?* In order to learn more about this, we look to the Hamiltonian.

Unfortunately, we cannot simply take the eigenstate with lowest energy to be our vacuum anymore since H is no longer time independent. We choose instead to look at a Hamiltonian at a specific time, η_0 . Then we need to find the mode functions associated to the minimization of the expectation value of H in the instantaneous vacuum, $\langle 0_v | H(\eta_0) | 0_v \rangle$, and the minimization of this value correspond to finding the eigenvector with the smallest eigenvalue[5]. In order to achieve this, we begin with our solution

$$\chi(x) = \frac{1}{\sqrt{2}} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} (e^{i\mathbf{x} \cdot \mathbf{k}} \dot{v}_{\mathbf{k}}^*(\eta) a_{\mathbf{k}}^- + e^{-i\mathbf{x} \cdot \mathbf{k}} \dot{v}_{\mathbf{k}}(\eta) a_{\mathbf{k}}^+)$$

where nothing is assumed to be isotropic, and plug it into our Hamiltonian

$$H(\eta) = \frac{1}{2} \int d^3 \mathbf{x} \left\{ \pi^2 + (\nabla \chi)^2 + m_{\text{eff}}^2(\eta) \chi^2 \right\}$$

where

$$\pi = \partial_\eta \chi$$

Our result is [6]

$$H(\eta) = \frac{1}{4} \int d^3 \mathbf{k} \left\{ a_{\mathbf{k}}^- a_{-\mathbf{k}}^- F_{\mathbf{k}}^* a_{\mathbf{k}}^+ a_{-\mathbf{k}}^+ F_{\mathbf{k}} + (2a_{\mathbf{k}}^+ a_{\mathbf{k}}^- + \delta(0)) E_{\mathbf{k}} \right\}$$

with

$$F_{\mathbf{k}} = (\dot{v}_{\mathbf{k}})^2 + \omega_k^2 v_{\mathbf{k}}^2, \quad E_{\mathbf{k}} = |\dot{v}_{\mathbf{k}}|^2 + \omega_k^2 |v_{\mathbf{k}}|^2$$

Taking the vacuum expectation value of this Hamiltonian will remove the first two terms since each annihilates a vacuum on either the left or right [7]. Then we have

$$\langle 0_v | H(\eta_0) | 0_v \rangle = \frac{1}{4} \int d^3 \mathbf{k} \delta(0) E_{\mathbf{k}}$$

Thus, our energy density is just

$$\epsilon = \frac{1}{4} \int d^3 \mathbf{k} |\dot{v}_{\mathbf{k}}|^2 + \omega_k^2 |v_{\mathbf{k}}|^2$$

Our next step is to minimize ϵ with our choice of mode function $v_{\mathbf{k}}$. Since each $\epsilon_{\mathbf{k}}$ must be minimized individually we fix \mathbf{k} . So at time $\eta = \eta_0$ we have initial conditions

$$v_{\mathbf{k}}(\eta_0) = \zeta, \quad \dot{v}_{\mathbf{k}}(\eta_0) = \xi$$

By our normalization condition, we have that

$$\xi \zeta^* - \zeta \xi^* = 2i$$

and thus we now seek ξ and ζ to minimize $|\xi|^2 + \omega_k^2 |\zeta|^2$. If we have found such ζ and ξ , then we can multiply each by a phase factor $e^{i\lambda}$ with $\lambda \in \mathbb{R}$ with impunity. Furthermore, we can choose λ such that ζ is strictly real and can write $\xi = \xi_1 + i\xi_2, \xi_{1,2} \in \mathbb{R}$. Then,

$$\begin{aligned} \xi \zeta^* - \zeta \xi^* &= \xi \zeta - \zeta \xi^* = \zeta(\xi - \xi^*) = 2i \\ \rightarrow \zeta &= \frac{2i}{\xi - \xi^*} = \frac{1}{\xi_2} \\ \rightarrow \epsilon_k &= \xi_1^2 + \xi_2^2 + \frac{\omega_k^2(\eta_0)}{\xi_2^2} \end{aligned}$$

So if we assume $\omega_k^2 > 0$ then minimizing with respect to ξ_1 and ξ_2 gives

$$\xi_1 = 0, \quad \xi_2 = \sqrt{\omega_k(\eta_0)}$$

respectively. Now, our ICs take the form

$$v_{\mathbf{k}}(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \quad \dot{v}_{\mathbf{k}}(\eta_0) = i\sqrt{\omega_k(\eta_0)} = i\omega_k v_{\mathbf{k}}(\eta_0)$$

If we suppose $\omega_k^2 < 0$, then we find that ϵ_k has no minimum and instead varies between $-\infty$ and ∞ , and so in this case the instantaneous lowest energy vacuum simply doesn't exist.

So if we define our mode functions according to the above ICs, we have the creation and annihilation operators and vacuum well defined. Using these ICs in the Hamiltonian we derived beforehand, we find that

$$E_k = 2\omega_k, \quad F_k = 0$$

and our Hamiltonian reduces to just the diagonal part

$$H(\eta_0) = \int d^3\mathbf{k} \omega_k(\eta_0) \left(a_{\mathbf{k}}^+ a_{\mathbf{k}}^- + \frac{1}{2} \delta(0) \right)$$

Now our Hamiltonian is diagonal in the eigenbasis consisting of the vacuum state and its excited states. This is sometimes called the vacuum of instantaneous diagonalization, but won't be referred to as such in the current consideration.

Since the mode functions are the same for all $|\mathbf{k}| = k$ and the lowest energy eigenstate is also isotropic[8].

Since, in general, ω_k isn't constant, we cannot let the general solution to F_k ,

$$v_k(\eta) = C \exp\left(\pm i \int \omega_k(\eta) d\eta\right)$$

solve the problem. Therefore, a lowest energy eigenstate that holds for all time η is impossible.

Vacuum Ambiguity and Physical Interpretation

We've already seen the ambiguity arise from different-particle vacuum states, so now that we've found the instantaneous vacuum states, let's choose two and compare them. We choose two arbitrary times, η_1 and η_2 and denote their vacuum states $|0_{1,2}\rangle$. Recall,

$$\langle 0_1 | N^{(2)} | 0_1 \rangle = |\beta_k|^2 \delta(0)$$

and use this to take the η_1 -vacuum expectation value of our minimized Hamiltonian of η_2 :

$$\begin{aligned} \langle 0_1 | H(\eta_2) | 0_1 \rangle &= \int d^3\mathbf{k} \omega_k(\eta_2) \langle 0_1 | \left(a_{\mathbf{k}}^+ a_{\mathbf{k}}^- + \frac{1}{2} \delta(0) \right) | 0_1 \rangle \\ &= \int d^3\mathbf{k} \omega_k(\eta_2) \delta(0) \left(|\beta_k|^2 + \frac{1}{2} \right) \end{aligned}$$

We previously considered the same calculation where the instantaneous Hamiltonian time corresponded with the vacuum states, and in comparison we see that this value is, in fact, greater. Thus, in general, a vacuum at time η_1 is an excited state at time η_2 .

Physically, in Minkowski spacetime, the vacuum and particles are a consequence of decomposing the field into plane waves. A particle of momentum p has a wave packet with a spread of momentum Δp ; for a well-defined particle we have $\Delta p \ll p$, and we also constrain its wavelength: $\lambda \gg p^{-1}$. But the geometry of the curved spacetime can, without any loss of generality, vary quite greatly within a region on the order of λ , and in this case, plane waves are not suitable.

We see that the notion of a particle is meaningful iff the spacetime geometry is approximately Minkowskian on distance and times scales of order p^{-1} . In our original analyses, for $\omega_k^2 < 0$ we get exponentials as solutions and the harmonic oscillator analogy also breaks down. We can still find creation and annihilation operators in this case, but we find that issues arise, such as excited states with energy less than the lowest energy vacuum states. Again, we address the fact that $\omega_k^2 < 0$ causes many problems, and in this situation there is no concept of lowest energy vacuum state.

This entire problem we are analysing arises from the fact that our field ϕ couples to gravity. Einstein taught us that gravity is equivalent to an accelerated frame of reference, and we can see from this that to choose a true physical vacuum would be the equivalent of selecting a preferred frame of reference, a clear violation of the first postulate of special relativity[3]. We note here that, if it was not already clear, the concepts of vacua and particles are inherently messy in the presence of gravity.

We have discussed the fact that $\omega_k^2 < 0$ makes everything fall apart, but since $\omega_k^2 = k^2 + m_{\text{eff}}^2$, even if the mass is negative we can select a k such that $\omega_k^2 > 0$. We demand

$$k^2 > k_{\text{min}}^2 = -m_{\text{eff}}^2$$

in order to satisfy this condition, and find that our length scale on which this holds is $L_{\text{max}} \sim k_{\text{min}}^{-1}$. In cos-

mological situations, however, L_{\max} is generally larger than the observable universe, so we needn't concern ourselves with this.

The Adiabatic Regime and Schrödinger's Equation

It isn't uncommon to see situations where, for a slow and small change from η_1 to η_2 , one vacuum can have an infinite particle density from the other time. We define the area in which ω_k is a slowly changing function of η and consider the WKB approximation[9]. For our differential equation

$$v_k'' + \omega_k^2 v_k = 0$$

the WKB approximation gives approximate solutions

$$\hat{v}_k(\eta) = \frac{1}{\sqrt{\omega_k(\eta)}} \exp\left(i \int_{\eta_0}^{\eta} \omega_k(\eta') d\eta'\right)$$

where \hat{v}_k is an approximate solution. If ω_k is slowly changing then the relative change of ω_k during one oscillatory period $\Delta\eta = 2\pi/\omega_k$ is negligibly small:

$$\begin{aligned} \left| \frac{\omega_k(\eta + \Delta\eta) - \omega_k(\Delta\eta)}{\omega_k(\eta)} \right| &= \left| \frac{\omega_k(\eta + \Delta\eta) - \omega_k(\Delta\eta)}{\Delta\eta} \frac{\Delta\eta}{\omega_k(\eta)} \right| \\ &\approx \left| \frac{\omega_k'}{\omega_k} \Delta\eta \right| \\ &= 2\pi \left| \frac{\omega_k'}{\omega_k} \right| \ll 1 \end{aligned}$$

This is called the adiabatic condition and the regime is the η for which this holds. We note that, here, it is not necessary for ω_k to be approximately constant. We can also define the adiabatic vacuum,

$$v_k(\eta_0) = \frac{1}{\sqrt{\omega_k(\eta_0)}}, \quad \frac{dv_k}{d\eta} = \left(i\omega_k - \frac{\omega_k'}{2\omega_k} \right) \frac{1}{\sqrt{\omega_k}}$$

however this does not minimize the vacuum. While the value is only slightly higher than that of the vacuum, it is insufficiently precise for many calculations.

The last thing we consider is an analogy to quantum mechanics: under the transformation $\eta \rightarrow x$ and $\omega_k \rightarrow E - V$ our differential equation becomes

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0$$

which is just Schrödinger's equation. We can imagine an incident wave $\psi = \exp(-ipx)$ from large $x > 0$. Then,

upon hitting a potential, we get a transmitted and reflected wave, $\psi = A_{\pm} \exp(\pm ipx)$ with $|A_-|^2 + |A_+|^2 = 1$, which bears resemblance to our condition for the Bogolyubov coefficients. The wave function behaves like the incident wave for $x < x_2$ and $x > x_1$. It must be emphasized, however, that this is only mathematically similar and the situation at hand bears no physical resemblance to the Schrödinger situation.

APPENDICES

Mathematical Proof of Statement in

The statement mentioned is that *minimizing $x^T Ax$ is equivalent to finding the eigenvector x with the smallest eigenvalue*. We use Lagrange multipliers to aid us in our quest: the problem we seek to solve is minimizing $x^T Ax$ subject to the constraint $\|x\|_2 = 1$. We acquire our constraint from the fact that we are addressing a problem restricted to L_2 space, where the 2-norm must be finite, and in quantum mechanics we generally insist on a normalization condition that this is unity. Thus,

$$\begin{aligned} \mathcal{L}(x) &= x^T Ax - \lambda x^T x + \lambda = x^T (A - \lambda)x + \lambda \\ \rightarrow \nabla \mathcal{L} &= (A + A^T)x - 2\lambda x \\ &= (A + A^T - 2\lambda)x = 0 \end{aligned}$$

We let $A + A^T = 2A$ which follows from the fact that we are considering a Hermitian operator. Since we assume, without loss of generality, that $x \neq 0$, $A - \lambda$ is singular and thus $\det(A - \lambda) = 0$, which is just the characteristic equation to the eigenvalue equation $Ax = \lambda x$. This is a result of assuming A is positive definite, i.e. $Ax > 0 \forall x$, but since we can always shift a Hamiltonian by a constant value to achieve this, we have no qualms with this assumption.

If we now assume we have some x_0 that solves $\nabla x_0^T Ax_0 = 0$ such that $Ax_0 = \lambda x_0$ then x_0 minimizes $x^T Ax$. If we left-multiply our eigenvalue equation by x_0^T , we get

$$x_0^T Ax_0 = x_0^T \lambda x_0 = \lambda \|x_0\|_2^2 = \lambda$$

and since $x_0^T Ax_0$ is minimal, so is λ . This concludes the proof.

REFERENCES

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- [3] Lev Landau, Evgeny Lifshitz, *Classical Theory of Fields*. Reed Educational and Professional Publishing Ltd, 2005.
- [4] Michael E. Peskin, Daniel V Schroeder, *Quantum Field Theory*. Westview Press, 1995.
- [5] For a proof of this fact, see the appendix.
- [6] A more explicit calculation was left out but is relatively straightforward, if tedious.
- [7] Notice, that we have the diagonal part of the Hamiltonian in brackets and the interaction pieces connected to our F functions.
- [8] It is worth noting that in a particular situation with a preferred direction, such as an anisotropic external field, the mode functions will be anisotropic and the Bogolyubov coefficients (and thus rate of particle production) will depend on direction.
- [9] This is considered in depth in [2]