RELATIVITY AND COSMOLOGY I

Problem Set 5a 25 October 2013

1. Maxwell equations in arbitrary coordinate systems

1. Show that
$$\Gamma^{\mu}{}_{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\nu \sqrt{|g|}\,.$$

2. Show that the divergence of a vector field J^μ can be written as
$$\nabla_\mu J^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} J^\mu\right)\,.$$

3. Show that for an antisymmetric rank-2 tensor $F^{\mu\nu}$
$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu \left(\sqrt{|g|} F^{\mu\nu}\right)\,.$$

4. Use the previous steps to show that Maxwell's equations in an arbitrary reference frame imply the conservation law
$$\nabla_\mu J^\mu = 0\,.$$

5. Show that if V^μ vanishes in a hypersurface S that bounds a region V of a n-dimensional space, then
$$\int_V \left(\nabla_\mu V^\mu\right) \sqrt{|g|}\, d^n x = 0\,.$$

6. Show that if $F^{\mu\nu}$ vanishes in a hypersurface S that bounds a region V of a n-dimensional space, then
$$\int_V \left(\nabla_\mu F^{\mu\nu}\right) \sqrt{|g|}\, d^n x = 0\,.$$

Indication: Start by showing that for an arbitrary matrix M depending on x we can write
$$\frac{d}{dx}\det M = \det M \cdot \mathrm{tr}\left(M^{-1}\frac{d}{dx}M\right)\,.$$

2. Commutator of covariant derivatives

Compute the commutator of two covariant derivatives acting on a vector, without assuming that the connection is symmetric.
Indication: Express the result as the sum of two terms; one proportional to the vector itself and one proportional to its covariant derivative.

3. Parallel transport on the 2-sphere

Consider a 2-sphere with coordinates (θ, ϕ) and metric

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \ .$$

1. Show that the lines of constant longitude ($\phi =$ constant) are geodesics.
2. Show that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).
3. Which are the components of a vector with components $V^\mu = (1, 0)$ after parallel-transporting it once around a circumference of constant latitude? Has its length changed? And its direction?
4. Compute the quantities $\nabla_\theta \nabla_\phi U^\theta$ and $\nabla_\phi \nabla_\theta U^\theta$ with $U^\mu = (0, 1)$. Do the covariant derivatives commute?
5. Compute the Riemann tensor for the 2-sphere.

4. Two-dimensional surfaces

Choosing appropriate coordinates, write the metric of the following two-dimensional surfaces embedded in three-dimensional Euclidean space.

1. A cylinder.
2. A sphere.
3. A cone.
4. A torus.
5. A hyperboloid of one sheet $x^2 + y^2 = R^2 + z^2$.
6. A sheet of a hyperboloid of two sheets $z^2 = R^2 + x^2 + y^2$, $z > 0$.