RELATIVITY AND COSMOLOGY I

Problem Set 4 18 October 2013

1. Polar coordinates

Consider the transformation from Cartesian to polar coordinates in a 2-dimensional Euclidean space. Compute :

1. The Jacobian of the transformation.
2. The basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in terms of $\mathbf{e}_x$ and $\mathbf{e}_y$.
3. The infinitesimal displacements dr and $d\theta$ in terms of dx and dy.
4. The metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ in two different ways.
5. The line element ds^2.
6. The volume element dV.
7. The Christoffel symbols.
8. The affine connection. Interpret the result geometrically.
9. The divergence of a vector field $V^\mu{}_{;\mu}$.
10. The Laplacian of a scalar field $\phi^{;\mu}{}_{;\mu}$.
11. The geodesic equation and its solution. Interpret the result.

2. Covariant derivative of a covariant vector

Use the fact that for a scalar $\phi_{;\mu} = \phi_{,\mu}$ together with the definition of the covariant derivative of a contravariant vector

$$V^\mu{}_{;\nu} = V^\mu{}_{,\nu} + \Gamma^\mu{}_{\rho\nu} V^\rho \ ,$$

to determine the form of the covariant derivative of a covariant vector.

3. Transformation law for the connection

1. Deduce the transformation law for the affine connection $\Gamma^\rho{}_{\mu\nu}$ from the relationship

$$\frac{\partial \mathbf{e}_\mu}{\partial x^\nu} = \Gamma^\rho{}_{\mu\nu} \mathbf{e}_\rho \tag{1}$$

and the transformation law of the basis vectors. Does the result ring the bell ?

2. Show that the difference of two connections is a tensor.

4. Symmetric connection

1. Consider a scalar function ϕ. Show that

$$\phi_{;\mu\nu} - \phi_{;\nu\mu} = 0 \ ,$$

   if and only if the connection is symmetric.
2. Consider two vectors fields U^μ and V^μ. Show that

$$U^\mu V^\nu{}_{;\mu} - V^\mu U^\nu{}_{;\mu} = U^\mu V^\nu{}_{,\mu} - V^\mu U^\nu{}_{,\mu} \ ,$$

   if and only if the connection is symmetric.

5. Metric compatibility

1. Show that the covariant derivative of the Kronecker delta is zero.
2. Show that $\nabla_\rho g^{\mu\nu} = 0$.