CHAPTER 8

THE SCHWARZSCHILD-DROSTE SOLUTION

> As you see, the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas
>
> Schwarzschild's letter to Einstein during World War I

Most of the work done till now has been related to weak-field solutions of the Einstein equations. In this Chapter, we go a step forward an look for exact solutions. Given the non-linearity of the field equations and the associated difficulty in finding analytical solutions for arbitrary matter distributions, we will restrict ourselves to *vacuum solutions*. To determine our starting point, let me rewrite the Einstein equations $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T^M_{\mu\nu}$ in a much more convenient form. Multiplying by the inverse metric and taking the trace we obtain a relation between the Ricci scalar, the cosmological constant and the trace $T^M \equiv g^{\mu\nu}T^M_{\mu\nu}$ of the locally conserved energy-momentum tensor $T^M_{\mu\nu}$, namely

$$R^\mu{}_\mu - \frac{1}{2}R\delta^\mu{}_\mu + \Lambda\delta^\mu{}_\mu = \kappa^2 T^M \qquad \longrightarrow \qquad R = -\kappa^2 T^M + 4\Lambda\,. \tag{8.1}$$

Substituting back this result into the original Einstein equations we realize that they can be written as

$$R_{\mu\nu} = \kappa^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) + \Lambda g_{\mu\nu}\,. \tag{8.2}$$

Vacuum solutions ($T^M_{\mu\nu} = \Lambda = 0$) correspond then to solutions of the equation

$$R_{\mu\nu} = 0\,, \tag{8.3}$$

rather that to solutions of $G_{\mu\nu}$=0.

Vacuum solutions are not necessarily flat

Eq. (8.3) does *not* imply the vanishing of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$, which contains extra components.

The problem of finding a solution of this equation is further simplified in those cases in which the problem is highly symmetric. In what follows, we will look for *spherically symmetric solutions.*

8.1 A spherically symmetric ansatz

Consider the spacetime *outside* a spherically symmetric mass distribution, which can be static or not. A spacetime is said to posses a particular symmetry if the functional form of the metric under the action of such a symmetry is maintained. In particular, a spherically symmetric spacetime is a spacetime whose line element is invariant under rotations (or, if you want. a spacetime "with the symmetries of the sphere"). The only rotational invariants of the spacelike coordinates $\mathbf{x} = x^i$ and their differential are

$$\mathbf{x}\cdot\mathbf{x} \equiv r^2\,, \qquad d\mathbf{x}\cdot d\mathbf{x}\,, \qquad \mathbf{x}\cdot d\mathbf{x}\,. \tag{8.4}$$

The most general spatially isotropic metric that can be constructed with these elements takes the form

$$ds^2 = -a(t,r)dt^2 - 2b(t,r)dt\,(\mathbf{x}\cdot d\mathbf{x}) + c(t,r)\,(\mathbf{x}\cdot d\mathbf{x})^2 + d(t,r)d\mathbf{x}\cdot d\mathbf{x}\,, \tag{8.5}$$

with a, b, c and d some *arbitrary* functions of t and r. The required invariance under rotations suggests the use of spherical coordinates $\{r, \theta, \phi\}$. Performing the change of variables we realize that all the angular dependence in (8.5) is isolated in the $d\mathbf{x}\cdot d\mathbf{x}$ part

$$\mathbf{x}\cdot\mathbf{x} = r^2\,, \qquad \mathbf{x}\cdot d\mathbf{x} = rdr\,, \qquad d\mathbf{x}\cdot d\mathbf{x} = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,. \tag{8.6}$$

Substituting these expressions into (8.5) we arrive to the equivalent form

$$ds^2 = -a(t,r)dt^2 - 2b(t,r)rdtdr + c(t,r)r^2dr^2 + d(t,r)\left(dr^2 + r^2 d\Omega^2\right)\,, \tag{8.7}$$

where we have defined $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. Collecting terms together and defining some, still arbitrary, functions

$$A(t,r) \equiv a(t,r)\,, \qquad B(t,r) \equiv rb(t,r)\,, \qquad C(t,r) \equiv r^2 c(t,r) + d(t,r)\,, \qquad D(t,r) \equiv r^2 d(t,r)\,,$$

to take into account the extra factors of r in Eq. (8.7), we are left with

$$ds^2 = -A(t,r)dt^2 - 2B(t,r)dtdr + C(t,r)dr^2 + D(t,r)d\Omega^2\,. \tag{8.8}$$

The resulting metric can be further simplified by using the freedom in the choice of coordinates. For instance, we can define a new radial coordinate $\bar{r}^2 \equiv D(t,r)$ and eliminate r and dr in terms of $\bar{r}, t, d\bar{r}$ and dt. This gives rise to a big mess that changes the explicit form of the coefficients A, B, C to some new, but still arbitrary, coefficients A', B', C'

$$ds^2 = -A'(t,\bar{r})dt^2 - 2B'(t,\bar{r})dtd\bar{r} + C'(t,\bar{r})d\bar{r}^2 + \bar{r}^2 d\Omega^2\,. \tag{8.9}$$

The next thing we can do is to find some new coordinate time $\bar{t}(t,\bar{r})$ to get rid of the nasty term $dtdr$. To do that, let me define this new time as

$$d\bar{t} = \mu(t,\bar{r})\left[A'(t,\bar{r})dt + B'(t,\bar{r})d\bar{r}\right] = \partial_t\Psi(t,r)dt + \partial_r\Psi(t,r)dr\,, \tag{8.10}$$

where the new unknown *integrating factor* μ is determined by the condition that the second equality holds for some Ψ. In other words, we require $\mu(t,\bar{r})\left[A'(t,\bar{r})dt + B'(t,\bar{r})d\bar{r}\right]$ to be a total differential, so that the first equality makes sense.

Squaring Eq.(8.10)

$$d\bar{t}^2 = \mu^2\left(A'^2dt^2 + 2A'B'dtd\bar{r} + B'^2d\bar{r}^2\right) \tag{8.11}$$

and isolating the terms related to dt^2 and $dtd\bar{r}$, we get

$$A'dt^2 + 2B'dtd\bar{r} = \frac{1}{A'\mu^2}d\bar{t}^2 - \frac{B'^2}{A'}d\bar{r}^2 \,. \tag{8.12}$$

In terms of the new temporal coordinate $\bar{t}$ the cross term disappears and the antsatz (8.9) becomes diagonal

$$ds^2 = -\frac{1}{A'\mu^2}d\bar{t}^2 + \left(C' + \frac{B'^2}{A}\right)d\bar{r}^2 + \bar{r}^2 d\Omega^2 \,. \tag{8.13}$$

Since the functions of $\bar{t}$ and $\bar{r}$ in this expression are arbitrary we can collect them into some arbitrary new functions[1]

$$e^{2\alpha} \equiv \frac{1}{A'\mu^2}\,, \qquad e^{2\beta} \equiv C' + \frac{B'^2}{A'}\,, \tag{8.14}$$

and write

$$ds^2 = -e^{2\alpha(\bar{t},\bar{r})}d\bar{t}^2 + e^{2\beta(\bar{t},\bar{r})}d\bar{r}^2 + \bar{r}^2 d\Omega^2 \,. \tag{8.15}$$

Dropping the bars to maintain the notation as light as possible, we arrive to our first important result

$$ds^2 = -e^{2\alpha(t,r)}dt^2 + e^{2\beta(t,r)}d\bar{r}^2 + r^2 d\Omega^2 \,. \tag{8.16}$$

Just by using spherical symmetry and our freedom to change coordinates, we have been able to reduce the 10 functions in $g_{\mu\nu}$ to two functions of only two variables! Rather impressive.

8.2 Spherical symmetry and staticity

The unknown functions α and β can be determined by inserting the antsatz (8.16) into the vacuum Einstein equations (8.3). The first step in this procedure is to compute the metric connection $\Gamma^{\mu}{}_{\nu\sigma}$. The job is conceptually straightforward but rather tedious. Whatever the way you do it[2], you should obtain 12 non-vanishing components out of 40, namely

$$\begin{aligned}
&\Gamma^t{}_{tt} = \partial_t\alpha\,, && \Gamma^t{}_{tr} = \Gamma^t_{rt} = \partial_r\alpha\,, && \Gamma^t{}_{rr} = e^{2(\beta-\alpha)}\partial_t\beta\,,\\
&\Gamma^r{}_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha\,, && \Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \partial_t\beta\,, && \Gamma^r{}_{rr} = \partial_r\beta\,,\\
&\Gamma^r{}_{\theta\theta} = -re^{-2\beta}\,, && \Gamma^r{}_{\phi\phi} = \sin^2\theta\,\Gamma^r_{\theta\theta}\,, && \Gamma^\theta{}_{r\theta} = \Gamma^\theta_{\theta r} = 1/r\,,\\
&\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta\,, && \Gamma^\phi{}_{\theta\phi} = \Gamma^\phi_{\theta\phi} = \cot\theta\,, && \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = 1/r\,.
\end{aligned} \tag{8.17}$$

The non-vanishing components of the Riemann tensor associated to these Christoffel symbols are given by

$$\begin{aligned}
&R^t{}_{rtr} = e^{2(\beta-\alpha)}\left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + \left[\partial_r\alpha\partial_r\beta - \partial_r^2\alpha - (\partial_r\alpha)^2\right]\,,\\
&R^t{}_{\theta t\theta} = -re^{-2\beta}\partial_r\alpha\,, \qquad R^t{}_{\phi t\phi} = -re^{-2\beta}\sin^2\theta\;\partial_r\alpha\,, \qquad R^t{}_{\theta r\theta} = -re^{-2\alpha}\partial_t\beta\,,\\
&R^t{}_{\phi r\phi} = -re^{-2\alpha}\sin^2\theta\;\partial_t\beta\,, \qquad R^r{}_{\theta r\theta} = re^{-2\beta}\partial_r\beta\,, \qquad R^r{}_{\phi r\phi} = re^{-2\beta}\sin^2\theta\;\partial_r\beta\,,\\
&R^\theta{}_{\phi\theta\phi} = (1-e^{-2\beta})\sin^2\theta\,. \qquad .
\end{aligned} \tag{8.18}$$

[1] This exponential form is specially useful for writing compact expressions for the components of the metric connections and the Riemann tensor.

[2] The quicker way to get $\Gamma^{\mu}{}_{\nu\sigma}$ is by using the Lagrangian procedure for geodesics, but you can also use the brute force method and compute them via Eq. (4.62).

which, contracted, provide us with the non-vanishing components of the Ricci tensor

$$\begin{aligned} R_{tt} &= \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] + \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \tfrac{2}{r}\partial_r\alpha\right] e^{2(\alpha-\beta)}\,, \\ R_{rr} &= \left[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\right] e^{2(\beta-\alpha)} - \left[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \tfrac{2}{r}\partial_r\beta\right]\,, \\ R_{tr} &= \frac{2}{r}\partial_t\beta\,, \qquad R_{\theta\theta} = 1 + e^{-2\beta}[r(\partial_r\beta - \partial_r\alpha) - 1]\,, \qquad R_{\phi\phi} = R_{\theta\theta}\sin^2\theta\,. \end{aligned} \tag{8.19}$$

Understanding the result

The result (8.19) can be easily understood from simple symmetry considerations. Consider for instance the $R_{r\theta}$ component and note that the metric (8.16) is invariant under "reflections" in the θ and ϕ coordinates, i.e. $\theta \to -\theta$ and $\phi \to -\phi$. When $\theta \to -\theta$, the sign of $R_{r\theta}$ changes and we are force to have $R_{r\theta} = 0$. The same kind of argument can be applied to many components to get

$$R_{r\theta} = R_{r\phi} = R_{t\theta} = R_{t\phi} = R_{\theta\phi} = 0\,. \tag{8.20}$$

The relation between $R_{\phi\phi}$ and $R_{\theta\theta}$ can be also derived without performing the explicit computation. To see this, consider the coordinate transformation $(\theta,\phi) \to (\bar\theta,\bar\phi)$ and write the expression for the angular part of the line element in both coordinate systems

$$d\theta^2 + \sin^2\theta d\phi^2 = \left[\left(\frac{\partial\theta}{\partial\theta'}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2\right] d\theta'^2 + \ldots \tag{8.21}$$

The invariance of the line element under rotations implies the equality

$$\left(\frac{\partial\theta}{\partial\theta'}\right)^2 + \sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2 = 1\,. \tag{8.22}$$

Substituting this into the transformation law for the $R_{\theta\theta}$ component

$$R_{\theta'\theta'} = \left(\frac{\partial\theta}{\partial\theta'}\right)^2 R_{\theta\theta} + \left(\frac{\partial\phi}{\partial\theta'}\right)^2 R_{\phi\phi} \quad\longrightarrow\quad R_{\theta\theta} = \left(1 - \sin^2\theta\left(\frac{\partial\phi}{\partial\theta'}\right)^2\right) + \left(\frac{\partial\phi}{\partial\theta'}\right)^2 R_{\phi\phi}$$

and demanding $R_{\theta'\theta'} = R_{\theta\theta}$, we get the sought-for relation $R_{\phi\phi} = \sin^2 R_{\theta\theta}$.

The empty-space field equations are obtained by setting each of the components (8.19) equal to zero. These gives rise to 5 equations among which only 4 are useful since the $R_{\phi\phi}$ component simply repeats the information of the $R_{\theta\theta}$ component. Among these 4 equations, the simplest one is that associated to R_{tr}. A simple inspection of this equation reveals a very interesting property: the function β must be independent of time

$$R_{tr} = 0 \quad\longrightarrow\quad \partial_t\beta = 0 \quad\longrightarrow\quad \beta = \beta(r)\,. \tag{8.23}$$

Taking into account this result and performing the time derivative of the vacuum equation $R_{\theta\theta} = 0$, we get

$$\partial_t R_{\theta\theta} = 0 \quad\longrightarrow\quad \partial_t\partial_r\alpha = 0 \quad\longrightarrow\quad \alpha = \gamma(r) + \kappa(t)\,. \tag{8.24}$$

The coefficient $e^{2\alpha(r,t)}$ can be then splited into two pieces $e^{2\alpha(r,t)} = e^{2\gamma(r)}e^{2\kappa(t)}$. This allows us to perform an extra coordinate redefinition

$$\mathrm{d}t \to e^{-\kappa(t)}\mathrm{d}t\,, \qquad \gamma(r) \equiv \alpha(r)\,, \tag{8.25}$$

in Eq. (8.16) to obtain a much simpler line element

$$ds^2 = -e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\Omega^2\,, \tag{8.26}$$

specified by only two *time-independent* functions $\alpha(r)$ and $\beta(r)$. The resulting metric is static[3] even though we did not impose any requirement on the source apart from being spherically symmetric. The source could be as dynamical as a collapsing or a pulsating star and the metric outside the matter distribution would still take the form (8.26), as long as the collapse is symmetric. This result is in perfect agreement with our discussion on gravitational waves: if a spherically symmetric body undergoes pure radial pulsations, there is no quadrupole and there is no emission of gravitational waves.

> All vacuum solutions of the Einstein equations with $SO(3)$ symmetry are necessarily static.

8.3 The Schwarzschild-Droste solution

Thanks to symmetry, we are left with 3 equations of a single variable r for two unknows α and β. Let me rewrite them as

$$R_{tt} = +\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2\alpha'}{r}\right)e^{2(\alpha-\beta)} = 0\,, \tag{8.27}$$

$$R_{rr} = -\left(\alpha'' + \alpha'^2 - \alpha'\beta' - \frac{2\beta'}{r}\right) = 0\,, \tag{8.28}$$

$$R_{\theta\theta} = 1 - e^{-2\beta}\left(1 + r\alpha' - r\beta'\right) = 0\,, \tag{8.29}$$

with the prime denoting derivatives with respect to r. Note that the first two equations are rather similar. Multiplying the first one by $e^{-2(\alpha-\beta)}$ and adding it to the second we get

$$e^{2(\beta-\alpha)}R_{tt} + R_{rr} = \frac{2}{r}\left(\alpha' + \beta'\right) = 0 \quad\longrightarrow\quad \alpha' + \beta' = 0 \quad\longrightarrow\quad \alpha(r) + \beta(r) = \text{constant}\,. \tag{8.30}$$

The integration constant appearing in the previous expression can be always set to zero by simply performing a coordinate redefinition, allowing us to set $\alpha = -\beta$. Inserting this result into Eq. (8.29) we get

$$R_{\theta\theta} = 0 \quad\longrightarrow\quad (1 + 2r\alpha')\,e^{2\alpha} = 1 \quad\longrightarrow\quad \left(re^{2\alpha}\right)' = 1\,, \tag{8.31}$$

which can be easily integrated to obtain

$$re^{2\alpha} = r + C \quad\rightarrow\quad e^{2\alpha} = e^{-2\beta} = 1 + \frac{C}{r}\,, \tag{8.32}$$

or equivalently

$$ds^2 = -\left(1 + \frac{C}{r}\right)dt^2 + \left(1 + \frac{C}{r}\right)^{-1}dr^2 + r^2 d\Omega^2\,. \tag{8.33}$$

[3] A static spacetime is one in which

i) The components of $g_{\mu\nu}$ are independent of the timelike component x^0.

ii) The line element is invariant under the transformation $x^0 \to -x^0$.

If the second condition is not satisfied the spacetime is rather said to be *stationary*. A particular example of stationary metric is the one generated by a rotating star, where the change $x^0 \to -x^0$ changes the sense of rotation.

The obtained metric is *asymptotically flat*: it tends to the Minkowski metric when $r \to \infty$.

Birkhoff's theorem

Any solution of the vacuum Einstein equations with $SO(3)$ symmetry must be static and asymptotically flat.

The only thing left is to associate the constant C to some physical parameter. The most important use of a spherically symmetric vacuum solution is to represent the spacetime outside stars or planets. In that case, we would expect to recover the Newtonian limit

$$g_{00} = -\left(1 - \frac{2GM}{r}\right) , \qquad g_{rr} = \left(1 + \frac{2GM}{r}\right) , \tag{8.34}$$

at large r values. Comparing (8.34) with the $r \to \infty$ limit of the metric (8.32)

$$g_{00} = \left(1 + \frac{C}{r}\right) , \qquad g_{rr} = \left(1 - \frac{C}{r}\right) , \tag{8.35}$$

we get $C = -2GM$, which allows us to write the final and traditional expression for the so-called *Schwarzschild-Droste* metric[4]

$$ds^2 = -\left(1 - \frac{R_S}{r}\right) \mathrm{d}t^2 + \left(1 - \frac{R_S}{r}\right)^{-1} \mathrm{d}r^2 + r^2 d\Omega^2 \tag{8.36}$$

with

$$R_S \equiv \frac{2GM}{c^2} \simeq 3\mathrm{km} \left(\frac{M}{M_\odot}\right) , \tag{8.37}$$

the *Schwarzschild radius*[5].

Exercise

- Verify that (8.36) satisfies Eq. (8.29). Explain why this is guaranteed to happen even though we initially had three equations for two unknowns.

- Show that the Schwarzschild metric can be written in a form that makes explicit its isotropic character, namely

$$ds^2 = -\frac{\left(1 - \frac{R_S}{4\rho}\right)^2}{\left(1 + \frac{R_S}{4\rho}\right)^2} dt^2 + \left(1 + \frac{R_S}{4\rho}\right)^4 \left(dx^2 + dy^2 + dz^2\right) , \tag{8.38}$$

  with

$$\rho = \frac{1}{2}\left(r - GM + \sqrt{r^2 - 2GMr}\right) . \tag{8.39}$$

[4] Karl Schwarzschild found this exact solution in 1915 while serving in the German army on the Russian front during the World War I and died a year later from pemphigus, a painful autoimmune disease. An alternative derivation of this solution based on the Weyl method was presented by Droste around the same time but for some reason the physics community completely ignored it.

[5] We have momentarily restored the factors of c.

8.4 Measuring distances and times

Which is the physical meaning of the coordinates (t, r, θ, ϕ) appearing in the Schwarzschild-Droste solution? Although they provide a global reference frame for an observer making measurements at an infinite distance from the source (asymptotic flatness), not all of them represent physical quantities measured by arbitrary observers. While θ and ϕ have the same interpretation than the spherical angular coordinates in flat spacetime, the coordinate radius r and the coordinate time t cannot be generically interpreted as the physical radius or the physical time measured by a clock.

Physical quantities must be computed from the metric. The physical interval in the radial direction measured by an arbitrary local observer is given by the proper distance ($dt = d\theta = d\phi$)

$$ds = \left(1 - \frac{R_S}{r}\right)^{-1/2} dr\,, \tag{8.40}$$

while the time measured by an stationary clock at r ($dr = d\theta = d\phi = 0$) is given by the proper time interval

$$d\tau = \left(1 - \frac{R_S}{r}\right)^{1/2} dt\,. \tag{8.41}$$

Understanding the result

In the Schwarzschild metric, space is foliated by spheres S^2 of area $4\pi r^2$ separated by a proper distance $\left(1 - \frac{R_S}{r}\right)^{-1/2} dr$.

8.5 Visualizing Schwarzschild spacetime

A mental image of the Schwarzschild-Droste spacetime can be obtained by embedding a subset of it into a higher dimensional spacetime[6]. Since our solution is static and spherically symmetric, we can, without loss of generality, fix t =constant and $\theta = \pi/2$. This leaves us with a 2-dimensional surface

$$dX^2 = \left(1 - \frac{R_S}{r}\right)^{-1} dr^2 + r^2 d\phi^2 = f(r)^{-1} dr^2 + r^2 d\phi^2\,, \tag{8.42}$$

which can be easily embeded into the ordinary 3-dimensional Euclidean space

$$dX^2 = dz^2 + dr^2 + d\phi^2 = \left[1 + \left(\frac{dz(r)}{dr}\right)^2\right] dr^2 + r^2 d\phi^2\,. \tag{8.43}$$

The function $z(r)$ can be obtained by simply comparing (8.42) and (8.43)

$$1 + \left(\frac{dz(r)}{dr}\right)^2 = f(r)^{-1}\,, \tag{8.44}$$

and performing a trivial integration

$$z(r) = \int_0^r dr' \sqrt{\frac{1 - f(r')}{f(r')}} = 2\sqrt{R_S(r - R_S)} + \text{constant}\,. \tag{8.45}$$

The resulting embedding diagram is the Flamm's paraboloid shown in Fig. 8.5. The distances between circles on this surface are larger than just Δr, in clear agreement with the discussion presented in the previous section.

[6] Remember that such embedding diagrams can be misleading. For instance, a 2-dimensional cylinder embedded in 3-dimensional Euclidean space can seem to be curved even though it is *intrinsically* flat, $K = \kappa_1 \kappa_2 = 0$.

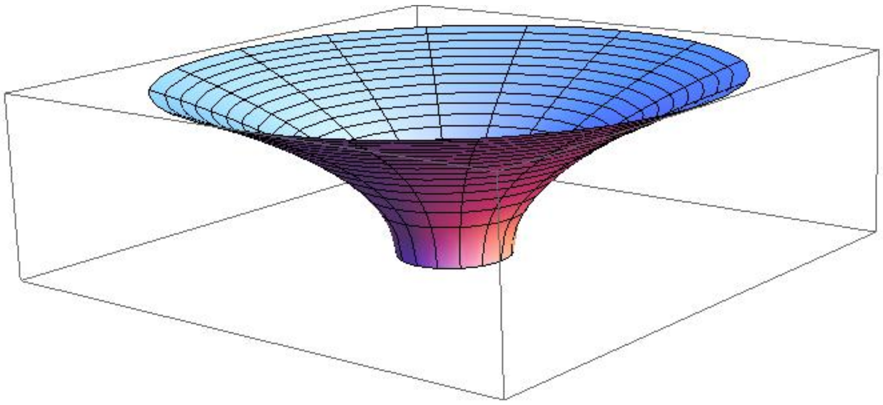
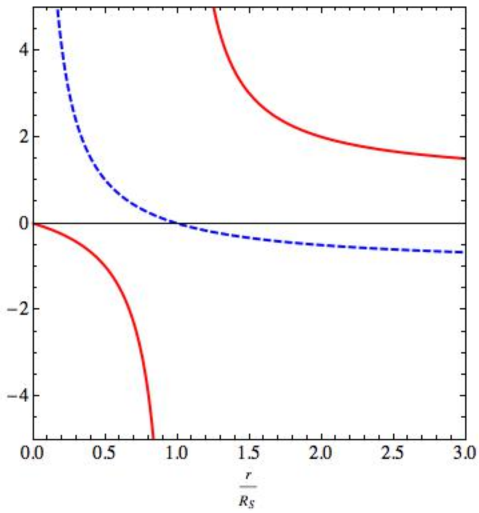


Figure 8.1: Embedding diagram for the Schwarzschild $(r - \phi)$ plane: Flamm's paraboloid

8.6 Apparent singularity

The line element (8.36) appears to contain two singularities, one at $r = 0$ coming from the g_{tt} component (blue dashed line) and another at $r = R_S$ coming from g_{rr} (red line).



Is this a problem? Not necessarily. In most of the astrophysical applications the typical size R of the source is much larger than the Schwarzschild radius (8.37)

$$\left.\frac{R_S}{R}\right|_{\oplus} \approx 10^{-9}\,, \qquad \left.\frac{R_S}{R}\right|_{\odot} \approx 10^{-6}\,, \qquad \left.\frac{R_S}{R}\right|_{\mathrm{NS}} \approx 10^{-1}\,. \tag{8.46}$$

This fact makes the singularities at $r = 0$ and $r = R_s$ completely irrelevant in most of the cases, since they lie in the interior of objects where the exterior Schwarzschild solution does not apply. Indeed, the problem disappears when one consider realistic interior solutions of the Einstein equations

$$ds^2 = -\left(1 - \frac{2GM(r)}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM(r)}{r}\right)^{-1}\mathrm{d}r^2 + r^2 d\Omega^2\,, \tag{8.47}$$

since the function $M(r)$ decreases faster than r and effectively kills all the above singularities.

We should worry and speculate about the singularities only in those cases in which the size of the object is such that the Schwarzschild-Droste solution applies all the way down to $r = R_S$. This kind of objects are called *black holes*. Even in that case the two singularities described above are not on equal footing. The metric coefficients in the line element (8.36) depend on the choice of a *particular coordinate system* and you should not extract any conclusion from them alone. Let me present an illustrative example.

A worked-out examples: Coordinates should not be trusted

Consider the completely regular and singularity-free Euclidean space in two dimensions

$$dX^2 = dx^2 + dy^2 \,, \tag{8.48}$$

and perform a general coordinate transformation to a new variable ρ defined through $x = 2\sqrt{\rho}$ to get

$$ds^2 = \frac{1}{\rho^2} d\rho^2 + dy^2 \,. \tag{8.49}$$

The metric appears to blow up at $\rho = 0$ even though we know that our space is, by construction, flat and free of singularities. The apparent singularity is a breakdown of our coordinate system at the point in which ρ becomes negative. It has nothing to do with a breakdown of the underlaying manifold!

In order to determine if we are dealing with some artifice of our coordinate system or with a true physical singularity, we cannot neither look to the curvature tensors alone, since their components are coordinate-dependent[7]. We should rather construct scalars out of the curvature tensors. If any the scalar blows up in a particular coordinate system, it will do in all of them. The simplest possibility would be to consider the Ricci scalar, R but we can also construct higher order scalars such as $R_{\mu\nu}R^{\mu\nu}$ $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$. For the particular case of the Schwarzschild-Droste metric, the first two quantities are not useful since are identically equal to zero. We are forced then to consider the square of the Riemann tensor, the so-called *Kretschmann scalar*. Taking into account the non-vanishing components of the Riemann tensor (8.18), we obtain

$$\mathcal{K} = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{12R_S^2}{r^6} \,, \tag{8.50}$$

which is a perfectly regular quantity at the Schwarzschild radius, but becomes infinity at $r = 0$. This last point is a real physical singularity! The singularity at $r = R_S$ is, on the other hand, just a pathology of the specific coordinate system used.

8.7 Geodesics in Schwarzschild metric

Let us study the motion of pointlike objects in our recently found Schwarzschild solution. To do that, let me consider the reparametrization invariant action (3.27)

$$S = \int L\, d\sigma = \frac{1}{2}\int d\sigma \left(e^{-1}(\sigma) g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - m^2 e(\sigma) \right) \,, \tag{8.51}$$

[7] They can catch singularities when going from one coordinate system to another through the transformation matrices $\partial \bar{x}^\mu / \partial x^\nu$.

in the massive ($e(\sigma) = 1/m$) and massless ($e(\sigma) = 1, m \to 0$) cases[8]

$$S_{\text{massive}} = \frac{1}{2}m \int d\sigma \left(g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - 1 \right) , \qquad S_{\text{massless}} = \frac{1}{2} \int d\sigma \left(g_{\mu\nu} \frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} \right) . \tag{8.52}$$

The geodesic equations for both actions can be directly written by taking into account the non-vanishing Christoffel symbols (8.17). Let's denote the derivatives with respect to the affine parameter σ by a dot. The explicit form[9] obtained by following this procedure turns out to be not very useful since the resulting equations are coupled

$$\ddot{t} = -\frac{R_S}{r(r - R_S)}\dot{r}\dot{t} , \tag{8.53}$$

$$\ddot{r} = -\frac{R_S(r - R_S)}{2r^3}\dot{t}^2 + \frac{R_S}{2r(r - R_S)}\dot{r}^2 - (r - R_S)\left(\dot{\theta} + \sin^2\theta\dot{\phi}^2\right) , \tag{8.54}$$

$$\ddot{\theta} = -\frac{2}{r}\dot{\theta}\dot{r} + \sin\theta\cos\theta\dot{\phi}^2 , \tag{8.55}$$

$$\ddot{\phi} = -\frac{2}{r}\dot{\phi}\dot{r} - 2\cot\theta\,\dot{\theta}\dot{\phi} . \tag{8.56}$$

Fortunately, our task can be greatly simplified by considering the symmetries of the Schwarzschild-Droste metric. Since (8.36) does not depend on the coordinates t and θ (they are *cyclic* coordinates in (8.52)), we have two conservation laws

$$\partial_t L = 0 \qquad \longrightarrow \qquad E = \left(1 - \frac{R_S}{r}\right)\dot{t} = \text{constant} , \tag{8.57}$$

$$\partial_\phi L = 0 \qquad \longrightarrow \qquad h = r^2 \sin^2\theta\dot{\phi} = \text{constant} , \tag{8.58}$$

with a clear physical interpretation. In the massless case, E and h are the relativistic energy and angular momentum that the particle would have at $r = \infty$. In the massive case, they are the relativistic energy and angular momentum *per unit mass*.

Exercise

Check this by taking the non-relativistic limit of (8.57) and (8.58) at the equatorial plane $\theta = \pi/2$.

Conservation of angular momentum means that the particle moves in a plane, which we can set to be the equatorial plane $\theta = \frac{\pi}{2}$ without loss of generality. Indeed, a simple inspection of Eq. (8.55) shows that if we consider a geodesic passing through a point on the equator $\theta = \frac{\pi}{2}$ and tangent to the equatorial plane $\dot{\theta} = 0$, we will always have $\ddot{\theta} = 0$ and $\dot{\theta} = 0$.

On top of the above symmetries, we have still a generic conservation law associated to the invariance of the action (8.51) under reparametrizations of the path $\sigma \to \sigma = f(\sigma)$ (cf. Section 3.5.1). This reads

$$\frac{d}{d\sigma}\left(g_{\mu\nu}u^\mu u^\nu\right) = 0 \qquad \longrightarrow \qquad g_{\mu\nu}u^\mu u^\nu = -\epsilon , \tag{8.59}$$

with $\epsilon = 1$ and 0 for massive and massless particles respectively. Expanding this equation[10]

$$-\left(1 - \frac{R_S}{r}\right)\dot{t}^2 + \left(1 - \frac{R_S}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = -\epsilon \tag{8.60}$$

[8]Remember that $\sigma = \tau$ in the massive case.
[9]Up to a global factor m in the massive case.
[10]Remember that $\theta = \pi/2$.

and plugging (8.57) and (8.58) we obtain a single equation for $r(\sigma)$

$$\frac{1}{2}\left(\frac{dr}{d\sigma}\right)^2 + V(r) = \mathcal{E}\,, \tag{8.61}$$

with

$$V(r) \equiv -\epsilon\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{r^3} \tag{8.62}$$

playing the role of an *exact* effective potential and

$$\mathcal{E} \equiv \frac{1}{2}\left(E^2 - \epsilon\right)\,. \tag{8.63}$$

Eq. (8.61) is structurally equivalent to that of a particle of unit mass and energy[11] $\mathcal{E}$ moving in an effective potential $V(r)$. It is interesting to compare the obtained potential with the Newtonian result

$$V_N(r) = -\frac{GM}{r} + \frac{h^2}{2r^2} \tag{8.64}$$

The first two terms in Eq. (8.62) are just the universal gravitational attraction and the centrifugal barrier that were already present in Newton's theory of gravity. The third term is new.

At sufficiently long distances, the extra contribution is rather small and does not significantly modify the Newtonian effective potential[12](cf. Fig. 8.4). The situation is completely different at short distances. The new term eventually dominates over the centrifugal barrier for small r and drives the potential to $-\infty$[13]. Let me analyze the massive and massless case separately.

Massive particles, $\epsilon = 1$, $\sigma = \tau$:

We can distinguish two cases:

- If $h^2 > 3R_S^2$, the potential displays both a maximum and a minimum at

$$\left.\frac{dV(r)}{dr}\right|_{\epsilon=1} = 0 \qquad \longrightarrow \qquad r_{\text{max,min}} = \frac{h^2}{R_S}\left[1 \pm \sqrt{1 - 3\left(\frac{R_S}{h}\right)^2}\,\right]\,, \tag{8.65}$$

  We have then four possibilities depending of the relation between the effective energy of the particle and the potential (cf. Fig. ??):

  1. Circular orbits: If $\mathcal{E} = V(r_{\text{max}})$ or $\mathcal{E} = V(r_{\text{min}})$ the particle describes an unstable or stable orbit respectively.
  2. Bound precessing orbits: If $0 > \mathcal{E} > V(r_{\text{min}})$ the particle is trapped into the potential and describes an elliptical orbit with shifting perihelion (see below).
  3. Scattering orbits: If $V(r_{\text{max}}) > \mathcal{E} > 0$ the particle bumps in the potential and retreats back to infinity.
  4. Plunging orbits: If $\mathcal{E} > V(r_{\text{max}})$ the particle sails over the top of the potential to finally spiral into the black hole.

[11] The true energy per unit mass in E but the effective potential for r rather responds to $\mathcal{E}$.
[12] The small correction will play however a central role! See next section.
[13] Note that the potential is always zero at $r = R_S$.

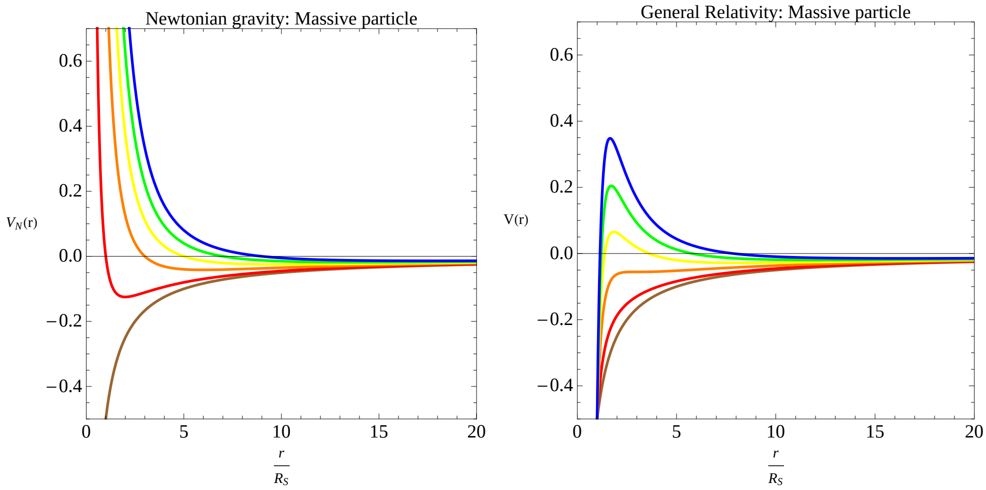


Figure 8.2: Effective potentials in Newtonian gravity and General Relativity for massive particles. Different lines correspond to $h^2/R_S^2 = 0, 1, 3, 5, 7, 9$ (from brown to blue). Note the change in the potential at the critical value $h^2/R_S^2 = 3$.

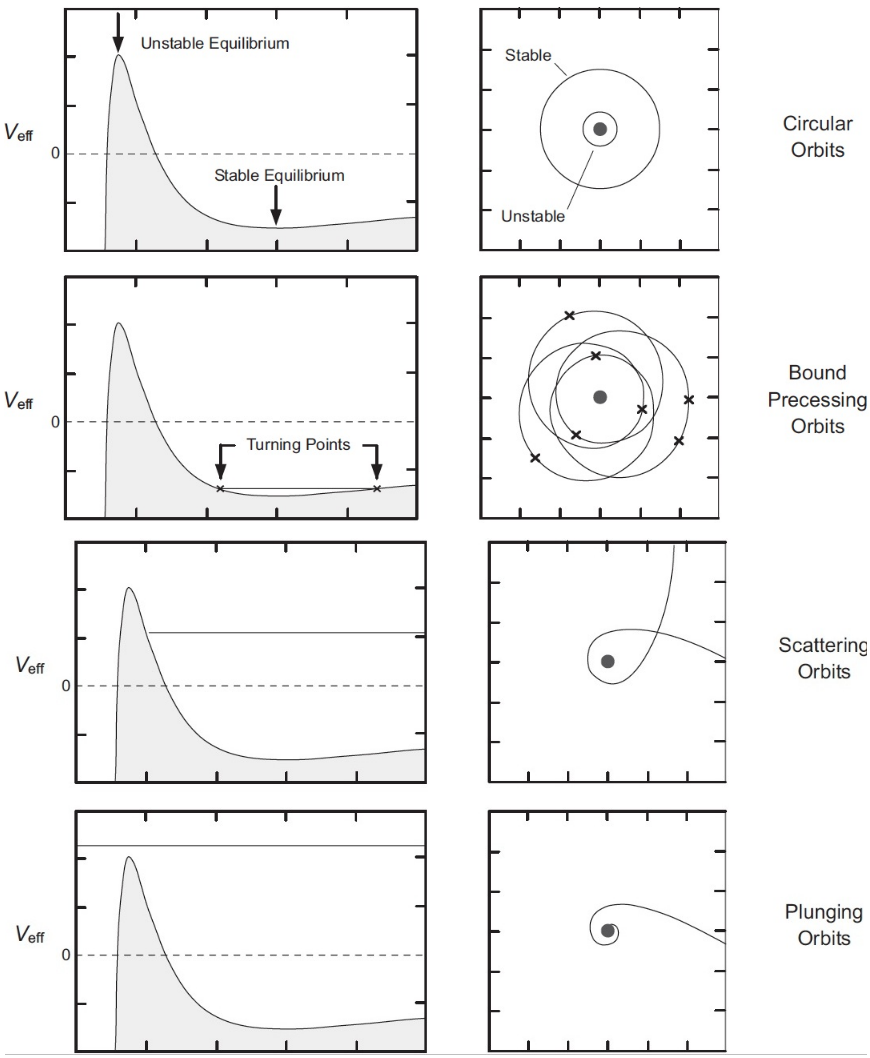


Figure 8.3: Orbits for massive particles in Schwarzschild-Droste geometry

Exercise
What happens with r_{max} and r_{min} when $h \to 0$? And when h decreases? Which is the minimal value of h and r allowing for a stable circular orbit?

- If $h^2 < 3R_S^2$ the centrifugal barrier disappears and the particle has no other option but to spiral into the singularity. Consider for clarity the limiting case $h = 0$ in which the particle follows a radial trajectory. In this case, the radial equation of motion (8.61) becomes[14]

$$\frac{dr}{d\tau} = \pm \left(\frac{R_S}{r}\right)^{1/2} \qquad \rightarrow \qquad \int \sqrt{r}dr = -R_S^{1/2} \int d\tau \, . \tag{8.66}$$

Integrating the previous equation we get

$$\tau(r) = \frac{2}{3\sqrt{R_S}} \left(r_0^{3/2} - r^{3/2}\right) , \tag{8.67}$$

with $r_0 > r$ an integration constant fixing the initial value of the proper time to zero. The particle reaches the Schwarzschild radius in a finite proper time τ. The interval measured by an observer at rest at spatial infinite is however quite different. Indeed, it is infinite, as can be easily seen by evaluating

$$\frac{dr}{dt} = \frac{d\tau}{dt}\frac{dr}{d\tau} = -\left(1 - \frac{R_S}{r}\right)\left(\frac{R_S}{r}\right)^{1/2} \qquad \rightarrow \qquad \int dt = -R_S^{-1/2} \int \frac{r^{3/2} dr}{r - R_S} \tag{8.68}$$

at $r = R_S$. For the observer at infinite the particle appears to approach but never quite cross the horizon! This is just another indication that the Schwarzschild coordinates are flawed near $R = R_S$.

Exercise
What happens with t when the observer crosses the horizon?

Massless particles, $\epsilon = 0$:

The potential (8.62) with $\epsilon = 0$ displays a unique maximum for all values of h at

$$r_{\text{max}} = \frac{3}{2} R_S \, . \tag{8.69}$$

Thus, the motion of massless particles can be divided into three cases:

1. Circular orbit: If $\mathcal{E} = V(r_{\text{max}})$ the particle describes an unstable circular orbit.

2. Scattering orbits: If $V(r_{\text{max}}) > \mathcal{E}$ the particle bumps in the potential and retreats back to infinity (deflection of light).

3. Plunging orbits: If $\mathcal{E} > V(r_{\text{max}})$ the particle sails over the top of the potential to finally spiral into the black hole.

[14] Among the two signs in the square root we take the negative one, in such a way that we fall toward $r \to 0$

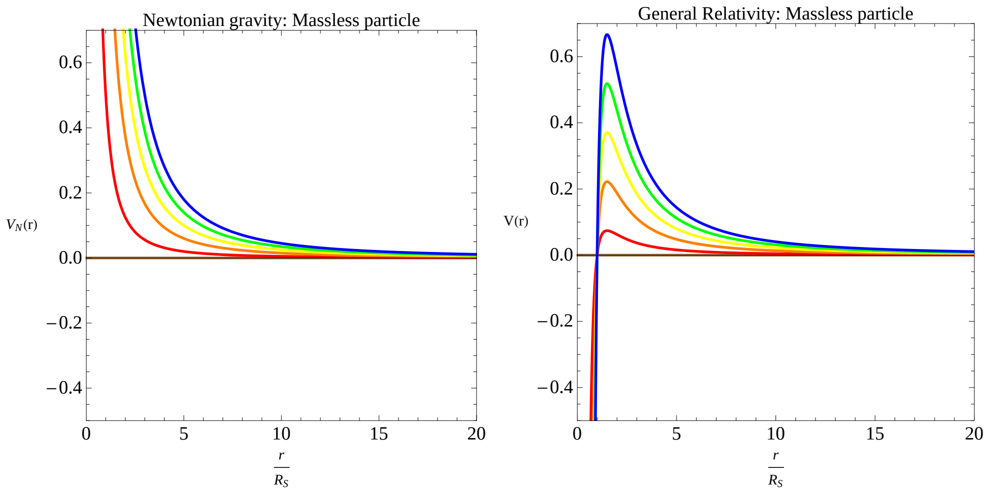


Figure 8.4: Effective potentials in Newtonian gravity and General Relativity for massless particles. Different lines correspond to $h^2/R_S^2 = 0, 1, 3, 5, 7, 9$ (from brown to blue).

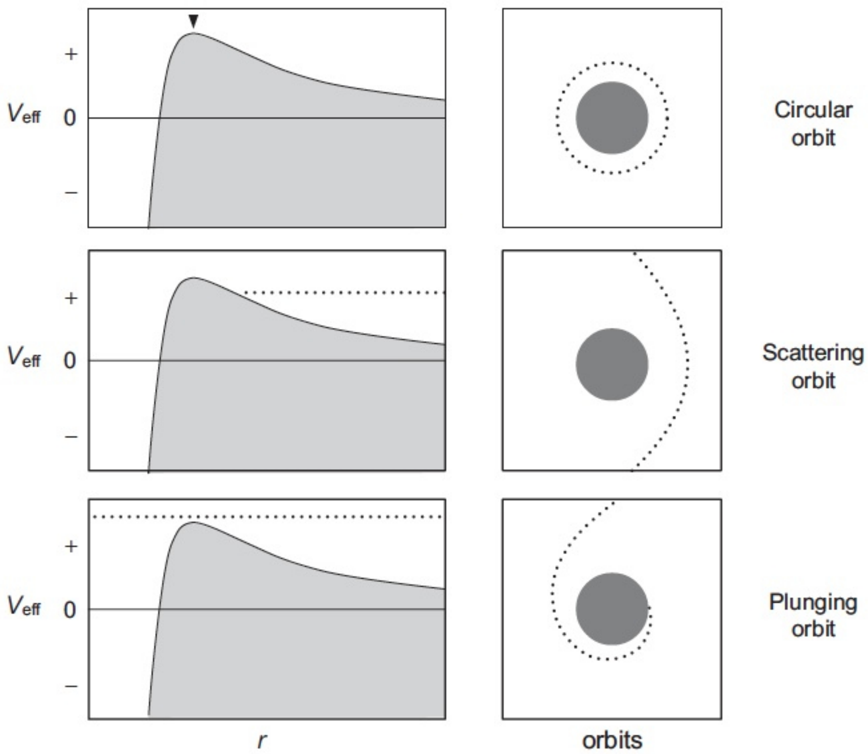


Figure 8.5: Orbits for massless particles in Schwarzschild-Droste geometry

8.8 Solving the radial equation

Let us determine the equation for the orbits described in the previous section. For doing that, we make use of Eq. (8.58) with $\theta = \pi/2$ and change the derivatives with respect to the affine parameter in Eq. (8.61) to derivatives with respect the angular variable ϕ

$$\frac{dr}{d\sigma} = \frac{dr}{d\phi}\frac{d\phi}{d\tau} = \frac{h}{r^2}\frac{dr}{d\phi}\,, \tag{8.70}$$

to obtain

$$\left(\frac{h}{r^2}\frac{dr}{d\phi}\right)^2 + \frac{h^2}{r^2} = \epsilon\frac{2GM}{r} + \frac{2GMh^2}{r^3} + 2\mathcal{E}\,. \tag{8.71}$$

The tricks to solve this kind of equation are well known. Let's perform a change of variable $u \equiv 1/r$ in (8.71)

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \epsilon\frac{2GMu}{h^2} + 2GMu^3 + \frac{2\mathcal{E}}{h^2}\,, \tag{8.72}$$

and derive the result with respect to ϕ. This gives rise to a second order differential equation of the form

$$\frac{d^2u}{d\phi^2} + u = \epsilon\frac{GM}{h^2} + 3GMu^2\,. \tag{8.73}$$

8.8.1 The massive case: Perihelion advance of Mercury

In the massive case $\epsilon = 1$ and (8.73) becomes

$$\frac{d^2u}{d\phi^2} + u = \frac{GM}{h^2} + 3GMu^2 \tag{8.74}$$

The resulting equation is extremely similar to the Newtonian equation of motion of a particle of mass m in the equatorial plane

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{GM}{h^2} \tag{8.75}$$

even though the interpretation of the radial variable r is completely different[15]. As you probably remember from your Classical Mechanics course, the general solution of (8.75) is a conic

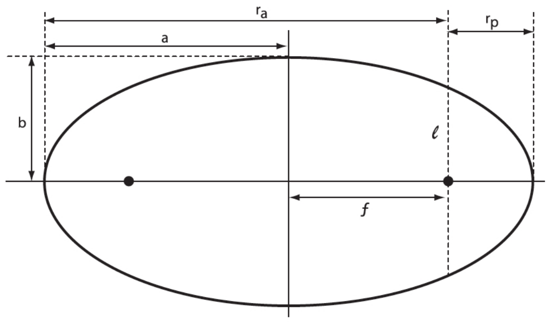
$$u_0 = \frac{GM}{h^2}(1 + e\cos\phi) \qquad \longrightarrow \qquad r_0 = \frac{a(1-e^2)}{(1 + e\cos\phi)} \tag{8.76}$$

with

$$a(1-e^2) = \frac{h^2}{GM}\,. \tag{8.77}$$

[15] In Newtonian gravity r is the radial *distance* from the mass while in the relativistic it is just a radial *coordinate* that can be only related to a distance through the metric.

Orbital Mumbo Jumbo



- **a=semi-major axis**: 1/2 of the long axis of the ellipse.
- **b=semi-minor axis**: 1/2 of the short axis of the ellipse.
- **e=eccentricity** : It characterizes the deviation of the ellipse from circular. When $e = 0$ the ellipse is a circle, when $e = 1$ the ellipse is a parabola. It is defined in terms of the semi major and semi minor axes a and b as

$$e = \sqrt{1 - \left(\frac{b}{a}\right)^2} . \tag{8.78}$$

- **f=focus**: The point over the semi-major axis at a distance $f = ae$ from the geometric center of the ellipse.
- **l=semi-latus rectum**: The distance $l = \frac{b^2}{a}$ from the focus to the ellipse along a line parallel to the semi-minor axis.
- r_p**=periapsis**: The distance $r_p = a(1-e)$ from the focus to the nearest point of approach of the ellipse.
- r_a**=apoapsis**: The distance $r_a = a(1+e)$ from the focus to the furthest point of approach of the ellipse.
- **The equation of the orbit:** It gives the distance to the orbiting body from the focus of the orbit as a function of the polar angle θ

$$r(\theta) = \frac{a(1-e^2)}{1 + e\cos\theta} . \tag{8.79}$$

If the gravitational field is sufficiently weak, Newtonian gravity alone is expected to provide a good approximation to the motion of massive particles in General Relativity. This suggest to treat to extra term $3GMu^2$ as a perturbation of top of the solution of Eq. (8.75). The perturbative solution of Eq. (8.74) can be determined by considering the antsatz

$$u = u_0 + \Delta u , \tag{8.80}$$

with u_0 given by (8.76). Inserting this into (8.74) we get

$$\frac{d^2 \Delta u}{d\phi^2} + \Delta u = A \left[\left(1 + \frac{e^2}{2}\right) + 2e\cos\phi + \frac{1}{2}e^2 \cos 2\phi \right] \tag{8.81}$$

with

$$A = \frac{3\,(GM)^3}{h^4} \tag{8.82}$$

a tiny parameter. To solve this equation let me notice two identities

$$\frac{d^2\phi}{d\phi^2}\left(\phi\sin\phi\right) + \phi\sin\phi = 2\cos\phi\,, \tag{8.83}$$

$$\frac{d^2}{d\phi^2}\left(\cos 2\phi\right) + \cos 2\phi = -3\cos 2\phi\,. \tag{8.84}$$

A direct comparison of (8.83) and (8.84) with (8.81) suggests the solution

$$\Delta u = A \left[\left(1 + \frac{e^2}{2}\right) - \frac{1}{6}\cos 2\phi + e\phi\sin\phi \right], \tag{8.85}$$

which can be checked by direct differentiation. The three terms in the square bracket are rather different. The first and the second one are just a constant and an oscillatory term around zero, both of them very small due to the constant A in front. The third one is different since it accumulates over successive orbits and gradually grows with time. Retaining only this last term we get

$$u = \frac{GM}{h^2}\left[1 + e\left(\cos\phi + \alpha\phi\sin\phi\right)\right] \tag{8.86}$$

which can be written in a more enlightening way

$$u \approx \frac{GM}{h^2}\left[1 + e\cos\left(1-\alpha\right)\phi\right] \tag{8.87}$$

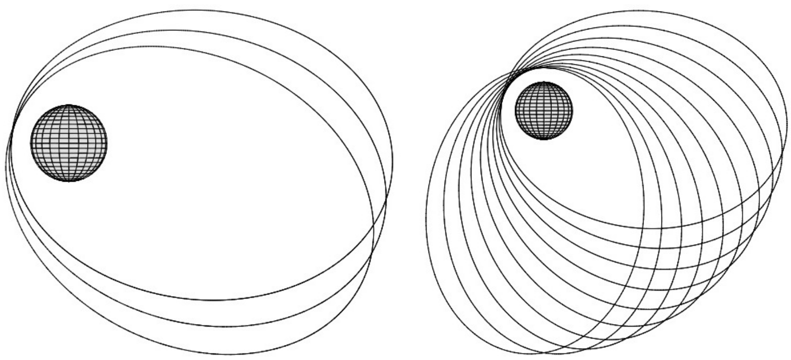
by taking into account that

$$\cos\left[\phi\left(1-\alpha\right)\right] = \cos\phi\cos\alpha\phi + \sin\phi\sin\alpha\phi \approx \cos\phi + \alpha\phi\sin\phi \tag{8.88}$$

for

$$\alpha \equiv \frac{3\,(GM)^2}{h^2} \ll 1\,. \tag{8.89}$$

The solution (8.87) shows that the orbit is still periodic, but with a period that is not longer 2π, but rather $2\pi(1-\alpha)$. The values of r repeats on cycles larger than 2π and the orbit precesses.



The advance of the perihelion in one revolution is

$$\Delta\phi = \frac{2\pi}{1-\alpha} - 2\pi \approx 2\phi\alpha = \frac{6\pi G^2 M^2}{h^2}\,, \tag{8.90}$$

which taking into account (8.77) can be written as[16] (note that we restore the c factors)

$$\Delta\phi = \frac{6G^2M^2}{h^2c^2} = \frac{6\pi GM}{a(1-e^2)c^2}\,. \tag{8.91}$$

Because it is a small effect, let's accumulate this over 100 years to get the observable quantity

$$\Delta\phi_{100} \equiv \frac{\Delta\phi}{T} \times \frac{100\,\text{years}}{\text{century}}\,, \tag{8.92}$$

with T the period of the orbit in years. In terms of observable orbits within the solar system, Mercury is the closest planet to the Sun, and so it should have the largest precession.

Object	Mass (10^{24} kg)	Mean Equatorial Radius (10^3 km)	Period (days)	Semimajor axis (10^8 km)	Eccentricity
Mercury	0.33010	2.4397	87.869	0.57909227	0.20563593
Venus	4.8673	6.0518	224.701	1.0820948	0.00677672
Earth	5.9722	6.3710	365.256	1.4959826	0.01671123
Mars	0.64169	3.3895	686.98	2.2794382	0.0933941
Jupiter	1898.1	69.911	4332.71	7.7834082	0.04838624
Saturn	568.32	58.232	10759.50	14.266664	0.05386179
Uranus	86.810	25.362	30685.00	28.706582	0.04725744
Neptune	102.41	24.622	60190.00	44.983964	0.00859048

Taking into account the values for Mercury's orbit, we obtain

$$\Delta\phi_{100} \approx 43.03'' \tag{8.93}$$

The major axis of Mercury precesses at a rate of 43 arcsecs per century. The observational results are in excellent agreement with General Relativity

Planet	Observed residual	GR prediction
Mercury	$(43.11 \pm 0.45)''$	$43.03''$
Venus	$(8.4 \pm 4.8)''$	$8.6''$
Earth	$(5.0 \pm 1.2)''$	3.8"

[16] The use of the expressions for the unperturbed solution is justified by the fact that we are looking to a very small quantity.

43 arcseconds and the end of the Newtonian empire

Newton's theory had been a very successful theory, extensively used by astronomers for centuries. It had predicted the return of comet Halley (1758) with an error of 33 days, the elliptical character of the recently discovered Uranus (1781) and even more surprisingly the location, mass and orbit parameters of Neptune, even before it was directly observed (1846). Leverrier discovered it just *with the point of his pen!*[a]; clearly an amazing proof of the universality of the gravitational interaction. Nevertheless, at the end of the 19th century there were still some caveats related to Mercury's orbit. As you should know the $1/r^2$ dependence of the Newton's force gives rise to elliptical trajectories on a plane, and the corresponding perihelion is *a priori* a fixed point[b]. However, different perturbations (due for instance to the presence of other massive objects in the Solar system, such as Jupiter, or to the quadrupole moment of the Sun), give rise to a perihelion advance, and therefore to an ellipse turning on the plane. Even when all those effects were taken into account there was a residual contribution to the shift. As pointed out by Leverrier and Newcomb at the end of the 19th, Mercury's perihelion precesses at a rate of 575$''$ per century, but only 532$''$ can be explained by the perturbations associated to the other planets. The remaining 43$''$ per century could not be accounted for by the Newtonian theory even when errors were taken into account. The observational problem was basically closed for everyone (apart from Leverrier[c]), but the theoretical problem would remain open till the introduction of General Relativity in 1915.

[a] Francois Arago 1786-1853.
[b] The Laplace-Runge-Lenz vector is conserved.
[c] He died believing that the history of the discovery of Neptune would repeat and a new planet with a mass enough to account for the 43$''$ per century would be encountered between the Sun and Mercury.

8.9 The massless case: Gravitational deflection of light

Let us consider now the massless case where $\epsilon = 0$ and (8.61) becomes

$$\frac{d^2u}{d\phi^2} + u = 3GMu^2 \,. \tag{8.94}$$

In the absence of the term $3GMu^2$, the previous solution reduces to the simple harmonic oscillator equation

$$\frac{d^2u_0}{d\phi^2} + u_0 = 0 \,, \tag{8.95}$$

whose solutions

$$u_0 = \frac{\sin\phi}{b} \,, \tag{8.96}$$

can be interpreted as straight lines passing at a distance b from the origin. Following a similar procedure to the one used in the previous section, we look for perturbative solutions of the form

$$u = u_0 + \Delta u = \frac{\sin\phi}{b} + \Delta u \tag{8.97}$$

with u_0 given by (8.96). Substituting (8.97) into (8.94) we get a linear equation in Δu

$$\frac{d^2\Delta u}{d\phi^2} + \Delta u = \frac{3GM}{b^2}\sin^2\phi \,, \tag{8.98}$$

whose solution is given by

$$\Delta u = \frac{3GM}{2b^2}\left(1 + \frac{1}{3}\cos(2\phi)\right) \,. \tag{8.99}$$

Adding this to the unperturbed solution we get

$$u = \frac{\sin\phi}{b} + \frac{3GM}{2b^2}\left(1 + \frac{1}{3}\cos(2\phi)\right), \tag{8.100}$$

which in the limit $r \to \infty$, $u \to 0$ and for small ϕ gives rise to

$$\phi \approx -\frac{2GM}{bc^2}. \tag{8.101}$$

The total deflection angle is twice this value

$$\Delta\phi = \frac{2R_s}{b} = \frac{4GM}{bc^2}. \tag{8.102}$$

For rays coming from a distant stars and grazing the surface of the Sun[17]

$$b \approx R_\odot = 6.96 \times 10^5\,\text{km} \qquad M_\odot = 2 \times 10^{30}\,\text{Kg} \tag{8.103}$$

we get

$$\Delta\phi = \frac{4GM_\odot}{c^2 R_\odot} = 1.75''. \tag{8.104}$$

Light paths so close to the Sun are of course not visible by day, but they become visible at the time of a total eclipse. Their position relative to the other background stars during the total eclipse appears shifted relative to the position in the usual night sky. This prediction of General Relativity was verified in 1919 just a few years later the formulation of the theory. Two separate groups led by Arthur Eddington and Andrew Crommelin moved to Guinea and Brazil to observe the total eclipse of May 29, 1919. They reported deflections of $(1.61 \pm 0.40)''$ and $(1.98 \pm 0.16)''$, in reasonable agreement with Einstein's prediction (8.104).

8.10 The post-Newtonian formalism

Nowdays, the agreement between theory and observation is at the level of a few parts in a thousand. The deviations from the General Relativity are usually parametrized in terms of the so-called post-Newtonian parameters β and γ measuring respectively the non-linearity in the superposition law for gravity and the spatial curvature produced by unit rest mass

$$ds^2 = -\left(1 - \frac{2GM}{r} + 2(\beta - \gamma)\frac{G^2M^2}{r^2}\right)dt^2 + \left(1 - \frac{2\gamma GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{8.105}$$

When this parameters are taken into account Eqs. (8.91) and (8.102) become respectively

$$\Delta\phi = \left(\frac{2 - \beta + 2\gamma}{3}\right)\frac{6\pi GM}{a(1-e^2)c^2}, \qquad \Delta\phi = \left(\frac{1+\gamma}{2}\right)\frac{4GM}{bc^2} \tag{8.106}$$

The General Relativity limit corresponds to $\gamma = \beta = 1$. Recent measurements provide values $\gamma = 0.9998 \pm 0.0003$ and $|2\gamma - \beta - 2| < 3 \times 10^{-3}$, in excellent agreement with GR.

[17] In this case, the effect is maximized and easier to observe.