CHAPTER 7

GRAVITATIONAL WAVES

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One of the most fascinating predictions of General Relativity is the existence of gravitational waves. Einstein theory of gravity abandons the Newtonian conception of space and time as a rigid structure in which the particles move. Spacetime is now alive and can curve, move and vibrate!

7.1 A bunch of questions

In this chapter we will try to answer the following questions

- How are gravitational waves generated?
- How do they propagate?
- Can we detect them? How?
- Why are they interesting?

Let me start by answering the simplest question, the second one.

7.2 Propagation in vacuum

The starting point for any discussion on gravitational waves is the time-dependent version of the linearized Einstein equations in the Lorenz gauge[1]

$$\Box \tilde{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,. \tag{7.1}$$

[1]The derivation of this equation was performed around a Minkowski background. A more general treatment for perturbations around an arbitrary background $g^{(0)}_{\mu\nu}$ exists. The so-called *Isaacson shortwave approximation* can be still applied in those cases in which the perturbative scale of the waves $h_{\mu\nu}$ is much smaller than the curvature scale of the background $g^{(0)}_{\mu\nu}$.

Consider the propagation of the perturbation $h_{\mu\nu}$ far away from the generating source. In this case, the energy-momentum tensor in Eq. (7.1) can be set to zero and we are left with the homogenous equation

$$\Box \tilde{h}_{\mu\nu} = 0 \,. \tag{7.2}$$

The resulting vacuum case is quite particular since it still contains a *residual* gauge freedom on top the Lorenz condition

$$\partial^\nu \tilde{h}_{\mu\nu} = 0 \,. \tag{7.3}$$

Having a look to Eqs. (6.62) and (6.63)

$$\tilde{h}^{\rm new}_{\mu\nu} = \tilde{h}^{\rm old}_{\mu\nu} - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho\right) \,, \tag{7.4}$$

$$\partial^\nu \tilde{h}^{\rm new}_{\mu\nu} = \partial^\nu \tilde{h}^{\rm old}_{\mu\nu} - \Box \xi_\mu \,, \tag{7.5}$$

we realize that we can still make an infinitesimal coordinate transformation $x^\mu \to x^\mu + \xi^\mu$ with $\Box\xi_\mu = 0$ without modifying the gauge condition (7.3). Indeed, if $\Box\xi_\mu$=0, we automatically have[2]

$$\Box \underbrace{\left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu}\partial_\rho \xi^\rho\right)}_{\xi_{\mu\nu}} = 0 \,, \tag{7.6}$$

meaning that we can always subtract the combination $\xi_{\mu\nu}$ from $\tilde{h}_{\mu\nu}$ in Eq. (7.2). The quantity $\xi_{\mu\nu}$ depends of 4-arbitrary functions ξ_μ, which can be chosen at will to impose 4 extra conditions on the perturbation $\tilde{h}_{\mu\nu}$. In particular, we can take ξ_0 and ξ_i in such a way that $\tilde{h} = 0$ and $\tilde{h}_{0i} = 0$. The condition of vanishing trace $\tilde{h} = 0$ erases the distinction between the perturbation and its trace reverse

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} \,. \tag{7.7}$$

On the other hand, the condition $\tilde{h}_{0i} = h_{0i} = 0$ applied the $\mu = 0$ component of the Lorenz gauge $\partial^\nu \tilde{h}_{\mu\nu} = 0$ implies that h_{00} is constant in time

$$\partial^0 \tilde{h}_{00} + \partial^i \tilde{h}_{0i} = \partial^0 h_{00} + \partial^i h_{0i} = 0 \,, \qquad \longrightarrow \qquad \partial^0 h_{00} = 0 \,. \tag{7.8}$$

This component corresponds to the static part of the gravitational interaction, i.e to the Newtonian potential of the source which gave rise to the gravitational wave. The gravitational wave itself is the time-dependent part. As far as gravitational waves are concerned, the condition $\partial^0 h_{00} = 0$ really means $h_{00} = 0$.

The discussion presented above defines the so-called *transverse-traceless* (TT) or *radiation gauge*

$$h_{0\mu} = 0 \,, \qquad h^i{}_i = 0 \,, \qquad \partial^j h_{ij} = 0 \,, \tag{7.9}$$

which completely fix all the local ambiguities and leaves us with $10-4-4 = 2$ degrees of freedom, the physical ones. The existence of such a gauge is guaranteed as long as there are no sources. Although, inside the source we are still allowed to perform a coordinate transformation with $\Box\xi_\mu = 0$ (or equivalently $\Box\xi_{\mu\nu}$=0) on top of the Lorenz gauge, we cannot set to zero any further component in $\tilde{h}_{\mu\nu}$, since $\Box\tilde{h}_{\mu\nu} \neq 0$. The situation is completely analogue to what happens in Classical Electrodynamics. Maxwell equations can be always reduced to the form $\Box A^\mu = J^\mu$ by imposing the Lorenz gauge condition $\partial_\mu A^\mu = 0$. Once there, we have still the freedom to implement a residual gauge transformation $A_\mu \longrightarrow A_\mu + \partial_\mu \xi$ with ξ satisfying the condition $\Box\xi = 0$. In the absence of sources, the function ξ can be used to get rid of one of the components in A^μ, let's say A^0. The Lorenz gauge reduces in this case to a transversality condition on A^i, namely $\partial_i A^i = 0$ and we are left with $4-1-1 = 2$ polarizations. If instead $j^0 \neq 0$, we have $\Box A^0 \neq 0$ and there is no choice of ξ able to satisfy simultaneously $\Box\xi = 0$ and $A^0 = 0$.

[2] The flat d'Alambertian $\Box$ commutes with ∂^μ.

7.2.1 Plane wave solutions

Eq. (7.2) admits a planar wave solution[3] of the form[4]

$$\tilde{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma} = A_{\mu\nu} e^{ik_i x^i} e^{-i\omega t} \,, \tag{7.10}$$

with $A_{\mu\nu}$ a symmetric rank-2 tensor called *polarization tensor* and $k^\mu = (\omega, \mathbf{k})$ a wave 4-vector satisfying the normalization condition[5]

$$\Box \tilde{h}_{\mu\nu} = \eta^{\rho\sigma}\partial_\rho \partial_\sigma \tilde{h}_{\mu\nu} = -k_\sigma k^\sigma \tilde{h}_{\mu\nu} = 0 \,, \qquad \longrightarrow \qquad k_\sigma k^\sigma = 0 \,. \tag{7.11}$$

Since k^σ is a null 4-vector, the dispersion relation takes the form $\omega = \mathbf{k}$; and gravitational perturbations propagate at the speed of light. The tranverse-traceless gauge (7.2) translates into the following restrictions on the components of the symmetric rank-2 tensor $A_{\mu\nu}$

$$A_{0i} = 0 \,, \qquad A^i{}_i = 0 \,, \qquad k^j A_{ij} = 0 \,. \tag{7.12}$$

To clarify our findings, let me consider a particular. Imagine a wave propagating in the z-direction. In this case, $k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega)$ and $A_{3i} = 0$, leaving as with only 4 non-vanishing components, namely $A_{11}, A_{12}, A_{21}, A_{22}$. Since A_{ij} is also symmetric and traceless, these components must satisfy $A_{11} = -A_{22} \equiv h_+$ and $A_{12} = A_{21} \equiv h_\times$, with h_+ and $h_\times$ the so-called *"plus" and "cross" polarizations*. Written in matricial form the coefficient $A_{\mu\nu}$ in the tranverse-traceless gauge takes the form

$$A^{\mathrm{TT}}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{7.13}$$

	Linearized Gravity		Electromagnetism
Plane wave solution	$\tilde{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\sigma x^\sigma}$		$A_\mu = a_\mu e^{ik_\sigma x^\sigma}$
Lorenz gauge	$k_\sigma k^\sigma = 0$		$k_\sigma k^\sigma = 0$
	$k^\mu A_{\mu\nu} = 0$		$k^\mu a_\mu = 0$
	$h_{00} = 0$	$h_{i0} = 0$	$A_0 = 0$
TT gauge	$\partial_i h^{ij} = 0$	$h^i{}_i = 0$	$\partial_i A^i = 0$
	$h_{ij} = h^{\mathrm{TT}}_{ij}$		$A_i = A^T_i$
	symmetric, transverse and traceless		transverse

7.3 Interaction of gravitational waves with matter

Once we have learned how to describe the propagation of gravitational waves, the next step is to discuss their interactions with matter, or in others words, the way of detecting them. Although it

[3] This is just the paradigmatic case. In the linear theory, we are always allowed to build an arbitrary wave-like solution by simply considering a superposition of these plane waves.

[4] The real part of the complex-valued expression (7.10) is assumed to be taken at the end of the computation, as usual.

[5] The components $A_{\mu\nu}$ are assumed to be constant.

might seem natural to think that we can learn something interesting by considering the geodesic equation

$$\frac{du^\mu}{d\tau} + \Gamma^\rho_{\mu\nu} u^\mu u^\nu = 0 \tag{7.14}$$

for a test particle in the gravitational field of the wave, this is not the case. To see this, consider our test particle to be at rest, $u^\mu = (1, 0, 0, 0)$, at an initial time, let's say, $\tau = \tau_0$. Evaluating the geodesic equation (7.14) at this time we get

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu_{00}\Big|_{\tau=\tau_0} = \frac{1}{2}\left(2\partial_0 h^{\rm TT}_{0i} - \partial_i h^{\rm TT}_{00}\right)\Big|_{\tau=\tau_0} , \tag{7.15}$$

which is identically zero since both h_{0i} and h_{00} are zero in the transverse-traceless gauge. The particle does not seem to experience any acceleration, it completely ignores the wave! Does this mean that gravitational waves have no effect in matter? Certainly not! It simply reflects the fact the Riemannian spacetime is locally flat at any given point.

Gauge freedom in General Relativity

In General Relativity, *gauge freedom* means *freedom to choose the coordinates*. The transverse-traceless gauge is a choice of frame which *moves with the particle* at the lowest order of approximation. The coordinates stretch themselves, in response to the arrival of the wave, in such a way that the position of the free test mass initially at rest does not change.

To detect gravitational waves we must go beyond a single point in spacetime and explore its neighborhood. Consider the wave (7.13) passing through a ring of test particles in $x - y$ plane. Let's denote by v^μ the distance of a test particles to the center of the ring and use the geodesic deviation equation

$$\frac{D^2 v^\mu}{d\tau^2} = \eta^{\mu\lambda} R_{\lambda\nu\rho\sigma} u^\nu u^\rho v^\sigma . \tag{7.16}$$

The linearized Riemann tensor

$$R_{\lambda\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho h_{\lambda\sigma} + \partial_\sigma\partial_\lambda h_{\nu\rho} - (\rho \leftrightarrow \sigma)\right) , \tag{7.17}$$

generated by the crossing gravitational wave is a gauge invariant quantity, meaning that we can compute it in any frame without affecting the result. Clearly the best choice is the TT gauge since the form of $h_{\mu\nu}$ in this frame is extremely simple. Assuming the particles to be moving slowly, $U^\nu \approx (1, 0, 0, 0)$, Eq. (7.16) becomes[6]

$$\frac{d^2 v^i}{dt^2} = R^i{}_{00j} v^j = \frac{1}{2}\frac{d^2 h^{\rm TT}_{ij}}{dt^2} v^j . \tag{7.18}$$

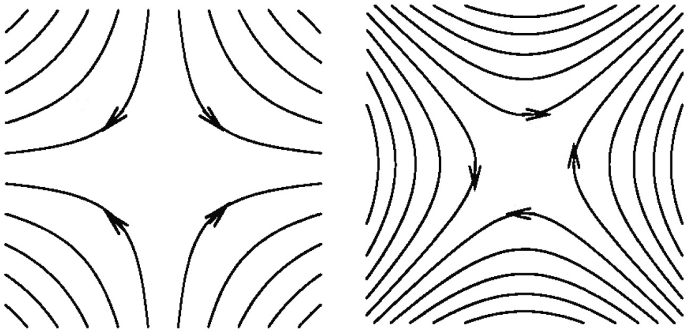
The resulting equation is extremely simple. The response of the particles can be understood in purely Newtonian terms, without any further reference to General Relativity. Since $h^{\rm TT}_{ij}$ is traceless, the effective Newtonian force per unit mass

$$F_i \equiv \frac{1}{2}\frac{d^2 h^{\rm TT}_{ij}}{dt^2} v^j , \tag{7.19}$$

is divergence free, $\partial^i F_i = 0$, meaning that there are no sources or sinks for the gravitational lines. Note also that, as in the electromagnetic case, only the transverse directions (v^x and v^y) to the wave propagation are affected (cf. Eq. (7.13)). If a particle is initially at $z = 0$, it will remain at $z = 0$.

[6] Note that, at leading order in $h_{\mu\nu}$, $\tau = t$.

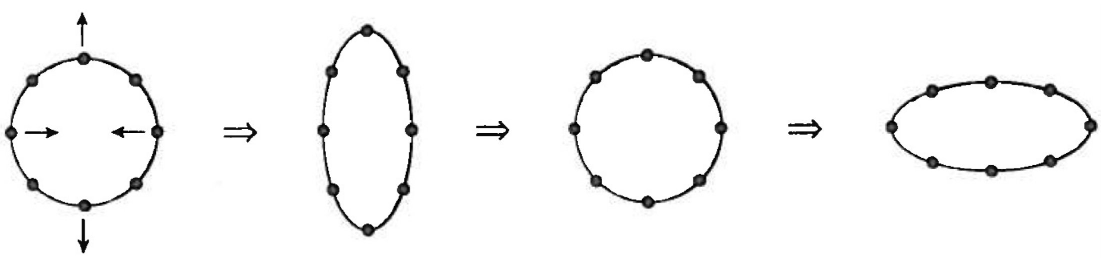
A pictorial representation of F^i can be obtained by drawing the lines of force the "plus" and "cross" polarizations



These lines are defined in such a way that at each point (x, y) they go in the direction of the force with a density proportional to the modulus of the force[7]. The effect of the components h_+ and $h_\times$ in the ring of particles is in clear agreement with the quadrupolar pattern displayed in the previous figure:

- **"Plus" polarization: $h_+ \neq 0$ and $h_\times = 0$:**

  In this case,
  $$\frac{d^2 v^x}{dt^2} = \frac{v^x}{2}\frac{d^2}{dt^2}(h_+ e^{ik_\sigma x^\sigma}), \qquad \frac{d^2 v^y}{dt^2} = -\frac{v^y}{2}\frac{d^2}{dt^2}(h_+ e^{ik_\sigma x^\sigma}), \tag{7.20}$$
  whose solution, to lowest order of accuracy, can be written as
  $$v^x = v_0^x + \frac{1}{2} h_+ e^{ik_\sigma x^\sigma} v_0^x \qquad v^y = v_0^y - \frac{1}{2} h_+ e^{ik_\sigma x^\sigma} v_0^y . \tag{7.21}$$
  with v_0^x and v_0^y staying for the initial separation of the particles in the x and y directions. A "+-polarized" wave makes the particles initially located in v_0^x and v_0^y bounce back and for in the x and y directions respectively. This fact, together with the 180° phase difference associated to the minus sign in Eq. (7.21), gives rise to the following pattern in the ring of particles

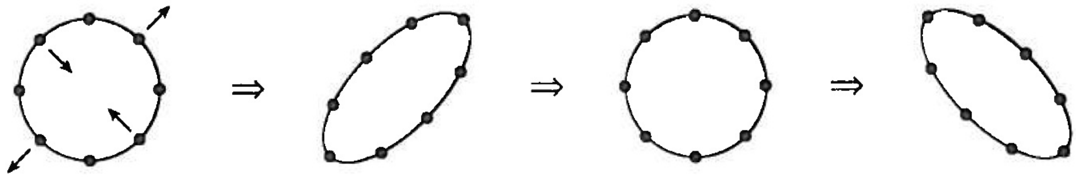


- **"Cross" polarization: $h_\times \neq 0$ and $h_+ = 0$:**

  In this case, the separation vector in a given direction depends also on the initial separation vector in the orthogonal direction
  $$v^x = v_0^x + \frac{1}{2} h_\times e^{ik_\sigma x^\sigma} v_0^y, \qquad v^y = v_0^y + \frac{1}{2} h_\times e^{ik_\sigma x^\sigma} v_0^x . \tag{7.22}$$

[7]Observe that the second figure can be obtained by rotating the first one 45°.

A "×-polarized" wave gives rise to a stretching and a squeezing along the $(2^{-1/2}, 2^{-1/2}, 0)$ and $(-2^{-1/2}, 2^{-1/2}, 0)$ directions. The ring of particles bounces back and for describing a cross shape.

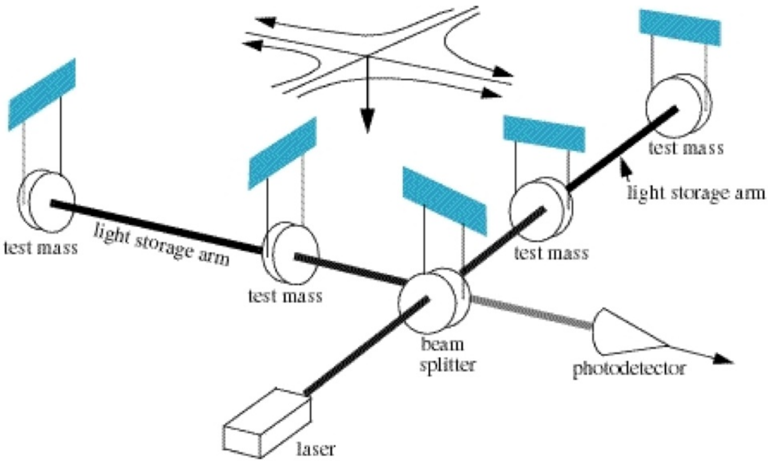


The components h_+ and $h_\times$ constitute the two independent linear polarizations of the gravitational wave and play a similar role to the vertical and horizontal polarization in electromagnetic waves. Different superpositions of these two modes can be always considered within the linear theory.

	Linearized Gravity	Electromagnetism
	$k^\mu = (\omega, 0, 0, \omega)$	$k^\mu = (\omega, 0, 0, \omega)$
Polarization	$A_{11} = -A_{22} \equiv h_+ \neq 0$	$a_1 \neq 0$
modes	$A_{12} = A_{21} \equiv h_\times \neq 0$	$a_2 \neq 0$
	$h_{\mathrm{R,L}} = \frac{1}{\sqrt{2}} (h_+ \pm i h_\times)$	$a_{\mathrm{R,L}} = \frac{1}{\sqrt{2}} (a_1 \pm i a_2)$

7.3.1 Laser interferometers

The most extended gravitational wave detectors are *laser based interferometric detectors*, whose basic operation can be summarized as follows: A laser beam is sent through a beam splitter and directed towards two very long resonant cavities. The light is reflected on mirrors at the end of the cavities and sent back to the beam splitter, which transmits half of each beam and reflects the other half. One part of each beam goes then back to the laser, while the other half-parts are combined to reach a photodetector in which the interference pattern is monitored. If a gravitational wave of amplitude h came out to pass trough the detector, its arm length will be periodically shorten in one direction and lengthen in the other, giving rise to a change in the interference pattern.



The total difference in length between the two arms can be derived[8] from Eq. (7.18)

$$\frac{\Delta L}{L} \sim h\,. \tag{7.23}$$

It is interesting to put some numbers. If we consider for instance the typical amplitude of gravitational waves emitted by a rotating binary system[9], $h \simeq 10^{-21}$, and a typical detector such as LIGO or Virgo with arm lengths of $3-4$ km, we get a change $\Delta L \simeq 10^{-16}$ cm.

7.4 The helicity of the graviton

In an hypothetical quantum theory of gravity, the gravitational waves presented in this Chapter would be quantized into particles satisfying the relativistic wave equation of a massless particle. The spin of the graviton can be inferred from the transformations properties of the classical field of the particle under rotations.

[8] We are implicitly assuming that the wave propagates orthogonally to the plane of the detector. In the general case, we get some angular coefficients of order 1.

[9] You will determine this number in the exercise session.

The Poincaré group has two physically interesting representations:

- **Massive representation**: These representations are characterized by the mass $m^2 = -p_\mu p^\mu$ and the spin s, which can take integer or half-integer values $s = 0, 1/2, 1, \ldots$. The representation with spin s has dimension $2s+1$. Example: A massive spin-1 particle has three-degrees of freedom.

- **Massless representation**: These representations are characterized by $p_\mu p^\mu = 0$ and a definite value of the *helicity*, which is defined as the projection of the total angular momentum (or the spin) in the direction of motion[a]

$$h = \mathbf{J} \cdot \mathbf{n} = (\mathbf{L} + \mathbf{S}) \cdot \mathbf{n} = \mathbf{S} \cdot \mathbf{n}\,. \tag{7.24}$$

  Under a rotation of angle θ around that direction a helicity eigenstate $|h\rangle$ transforms as

$$|h\rangle \longrightarrow e^{ih\theta}|\phi\rangle\,. \tag{7.25}$$

  There are always two helicity states $h = \pm s$, corresponding to the alignment or counter-alignment of the spin and the momentum.

[a]Unfortunately the traditional symbol h for the helicity coincides with some of the notations used in this chapter.

Applying a global rotation of angle θ

$$R_{ij} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7.26}$$

to our plane wave (7.13), we get

$$A^{\rm TT}_{ij} = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j A^{\rm TT}_{kl} \longrightarrow \begin{pmatrix} h_+\cos 2\theta + h_\times \sin 2\theta & h_\times \cos 2\theta - h_+ \sin 2\theta & 0 \\ h_\times \cos 2\theta - h_+ \sin 2\theta & -h_+\cos 2\theta - h_\times \sin 2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.27}$$

In the quantum theory the two polarization amplitudes h_+ and $h_\times$ become annihilation operators of gravitons and the circular polarization operator, $h_{\rm R,L} \equiv (h_+ \pm i h_\times)$, will transform as

$$h_{\rm R,L} \longrightarrow U h_{\rm R,L} U^\dagger \tag{7.28}$$

with $U = e^{iJ_3\theta/\hbar}$. Thus

$$h_{\rm R,L} \longrightarrow e^{\mp 2i\theta} h_{\rm R,L} \tag{7.29}$$

showing the spin of the graviton is 2, in unit of $\hbar$.

	Linearized Gravity	Electromagnetism
Helicity	$A_{ij} = \left(R^{-1}\right)^k{}_i \left(R^{-1}\right)^l{}_j A_{kl}$ $h_{\rm R,L} \longrightarrow e^{\mp 2i\theta} h_{\rm R,L}$	$a_i = \left(R^{-1}\right)^j{}_i a_j$ $a_{\rm R,L} \longrightarrow e^{\mp i\theta} a_{\rm R,L}$