CHAPTER 5

TIDAL FORCES AND CURVATURE

What are the differential laws which determine the Riemann metric (i.e. $g_{\mu\nu}$) itself?... The solution obviously needed invariant differential systems of the second order taken from $g_{\mu\nu}$. We soon saw that these had been already established by Riemann.

A. Einstein

In both Newtonian mechanics in the absence of gravity and Einstein's theory of Relativity, inertial frames are characterized by the absence of accelerations, which are absolute elements of the theory. If particles move in straight lines at constant speed the system is inertial. On the other hand, if the trajectory in spacetime is not a straight line the system must be accelerating. The situation is slightly different when gravity is taken into account. The equality between inertial and gravitational masses does not allow to *locally* distinguish the acceleration of a given reference frame from purely gravitational effects. Gravity can be *locally switched off* by properly choosing a local inertial frame associated to an observer in free-fall in the gravitational field. The word *locally* is fundamental, since the *global* behaviours of accelerations and gravity are completely different: while the true gravitational field vanishes at large distances, the apparent gravitational field in an accelerating frame takes a nonzero constant value at infinity. Real and apparent gravity can be distinguished by tracking the relative acceleration of nearby local inertial observers that appears due to the non-homogeneity of the gravitational field!

5.1 Gravity is a central force: Tides

Non-uniform gravitational fields are observable. Consider for instance two non-interacting particles falling towards the surface of the Earth (cf. Fig.5.1). Since the Earth is spherical in shape, both particles move towards the center of the Earth in such a way the separation between them decreases as they fall. The central character of the gravitational field gives rise to *tidal forces*. Let's put this into equations.

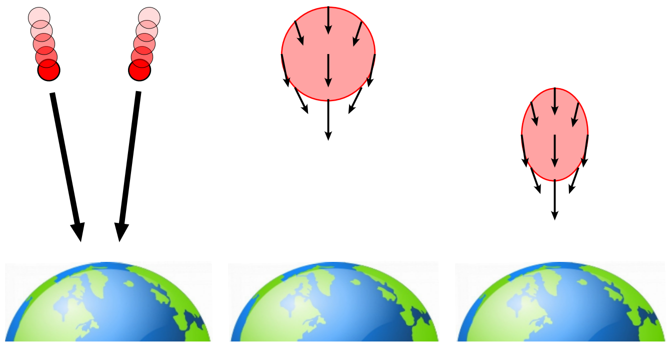

Figure 5.1: The effect of tidal forces.

In an inertial frame the equations of motion for the particles are given by the usual Newtonian expressions, namely

$$\frac{d^2x^i}{dt^2} = -\delta^{ik}\frac{\partial\Phi(x^j)}{\partial x^k}\,, \tag{5.1}$$

$$\frac{d^2(x^i+\xi^i)}{dt^2} = -\delta^{ik}\frac{\partial\Phi(x^j+\xi^j)}{\partial x^k}\,, \tag{5.2}$$

with ξ^i the separation vector between the two particles. For sufficiently small separations Eq. (5.2) can be Taylor expanded to linear order in ξ^i to obtain

$$\frac{d^2(x^i+\xi^i)}{dt^2} = -\delta^{ik}\left(\frac{\partial\Phi(x^i)}{\partial x^k} + \frac{\partial}{\partial x^j}\left(\frac{\partial\Phi(x^i)}{\partial x^k}\right)\xi^j + \dots\right)\,. \tag{5.3}$$

The *Newtonian deviation equation* for the separation vector ξ^i becomes therefore

$$\frac{d^2\xi^i}{dt^2} = -\delta^{ik}\left(\frac{\partial^2\Phi}{\partial x^k\partial x^j}\right)\xi^j\,. \tag{5.4}$$

The non-relativisitic *tidal tensor*

$$E^i{}_j \equiv \delta^{ik}\frac{\partial^2\Phi}{\partial x^k\partial x^j}\,, \tag{5.5}$$

determines the *tidal forces*, which tend to bring the particles together. This is the fundamental object for the description of gravity and not their individual accelerations $g_i = \partial_i\Phi$!

Exercise

Assume the tidal tensor $E^i{}_j$ to be reduced to diagonal form, as in the example below. Show that the components of that tensor cannot all have the same sign.

As a particular example, that will be useful in the future, consider two particles in the gravitational field of a spherically symmetric distribution of mass M, i.e $\Phi = -GM/r$. The tidal tensor (5.5) in this case becomes

$$E_{ij} = (\delta_{ij} - 3n_in_j)\frac{GM}{r^3}\,, \tag{5.6}$$

where $n^i \equiv x^i/r$ are the components of the unit vector in the radial direction. Writing explicitly the different components in polar coordinates we obtain

$$\frac{d^2\xi^r}{dt^2} = +\frac{2GM}{r^3}\xi^r\,, \qquad \frac{d^2\xi^\theta}{dt^2} = -\frac{GM}{r^3}\xi^\theta\,, \qquad \frac{d^2\xi^\phi}{dt^2} = -\frac{GM}{r^3}\xi^\phi\,. \tag{5.7}$$

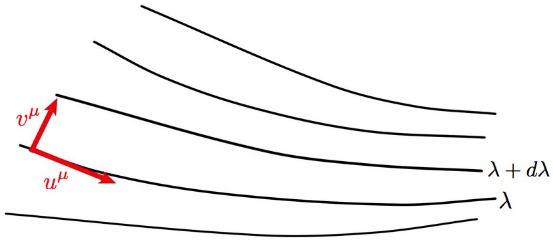

Figure 5.2: Bunch of geodesics classified by the value of λ.

Note the different signs: the object is stretched in the radial direction and compressed in the transverse directions. Tidal forces squeeze a sphere into an ellipsoid (cf. Fig.5.1).

Exercise
Assuming the water in the oceans to be in static equilibrium and taking into account the results of the previous example, estimate the height of the tides generated by the Moon.

Using the tidal tensor (5.5) we can write the equations governing the structure of Newtonian gravity in the following suggestive way

$$E^i{}_i = 4\pi G\rho \qquad \text{Poisson's equation} \tag{5.8}$$

$$\frac{d^2\xi^i}{dt^2} = -E^i{}_j\,\xi^j \qquad \text{Geodesic Deviation} \tag{5.9}$$

$$\left.\begin{aligned} E_{ij} &= E_{ji} \\ E^i{}_{[j,l]} &= 0 \end{aligned}\right\} \qquad \text{Bianchi Identities} \tag{5.10}$$

where the symbol $[j, l]$ stands for antisymmetrization in the corresponding indices, i.e.

$$E^i{}_{[j,l]} \equiv \frac{1}{2}\left(E^i{}_{j,l} - E^i{}_{l,j}\right) . \tag{5.11}$$

5.2 Geodesic deviation

Let us now study this issue taking into account the things that we learned in the previous chapter. Consider a bunch of geodesics $x^\mu(\sigma, \lambda)$ classified by the value of some parameter λ (cf. Fig. 5.2). Which is the requirement for having tidal forces? To answer this question, let me define two kinds of vectors (cf. Fig. 5.2): the tangent vector to the trajectory, $\frac{\partial x^\mu(\sigma,\lambda)}{\partial\sigma}$, that we will shortly denote by $u^\mu(\sigma, \lambda)$, and the derivative in the λ direction, $\frac{\partial x^\mu(\sigma,\lambda)}{\partial\lambda}$, that we will shortly denote by v^μ.

Taking Newtonian gravity as a guide, we expect the motion of the particles to be described by a second order differential equation involving the change of the separation vector v^μ along the path

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma \nabla_\sigma \left(u^\rho \nabla_\rho v^\mu\right) . \tag{5.12}$$

The right hand-side of this equation should contain the information about the true gravitational field. Using the relation[1]

$$v^\rho \nabla_\rho u^\mu = u^\rho \nabla_\rho v^\mu \,, \tag{5.13}$$

[1] It follows directly from the definition of the covariant derivatives and the relation $\partial u^\mu/\partial\lambda = \partial v^\mu/\partial\sigma$.

between the covariant derivatives of u^μ and v^μ, we get two pieces

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma \nabla_\sigma \left(u^\rho \nabla_\rho v^\mu \right) = u^\sigma \nabla_\sigma \left(v^\rho \nabla_\rho u^\mu \right) = u^\sigma \left(\nabla_\sigma v^\rho \right) \left(\nabla_\rho u^\mu \right) + u^\sigma v^\rho \nabla_\sigma \nabla_\rho u^\mu \, . \tag{5.14}$$

Changing the order of the covariant derivatives appearing in the first piece and using back Eq. (5.13) in the second piece, we obtain

$$\begin{aligned} \frac{D^2 v^\mu}{d\sigma^2} &= \underbrace{u^\sigma \left(\nabla_\sigma v^\rho \right)}_{v^\sigma (\nabla_\sigma u^\rho)} \left(\nabla_\rho u^\mu \right) + u^\sigma v^\rho \underbrace{\nabla_\sigma \nabla_\rho}_{\nabla_\rho \nabla_\sigma + [\nabla_\sigma, \nabla_\rho]} u^\mu \\ &= \underbrace{v^\sigma \left(\nabla_\sigma u^\rho \right) \left(\nabla_\rho u^\mu \right)}_{\sigma \leftrightarrow \rho} + u^\sigma v^\rho \nabla_\rho \nabla_\sigma u^\mu + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho \right] u^\mu \\ &= v^\rho \left(\nabla_\rho u^\sigma \right) \left(\nabla_\sigma u^\mu \right) + u^\sigma v^\rho \nabla_\rho \nabla_\sigma u^\mu + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho \right] u^\mu \\ &= v^\rho \nabla_\rho \left(u^\sigma \nabla_\sigma u^\mu \right) + u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho \right] u^\mu \, , \end{aligned} \tag{5.15}$$

where in the last steps we have simply performed some index relabelings and collected terms. The first term in the last line of (5.15) vanishes since, as we show in Section 4.6, the tangent vector to the trajectory is parallel transported along the geodesic, $u^\sigma \nabla_\sigma u^\mu = 0$. We are left therefore with a very compact expression

$$\frac{D^2 v^\mu}{d\sigma^2} = u^\sigma v^\rho \left[\nabla_\sigma, \nabla_\rho \right] u^\mu \, , \tag{5.16}$$

which hides however a big amount of work inside the commutator of the two covariant derivatives.

What we should expect

Before proceeding to the explicit computation of this commutator, let me anticipate what is gonna happen. Note that the commutator of two covariant derivatives acting on a scalar ϕ

$$[\nabla_\sigma, \nabla_\rho] \phi = \nabla_\sigma \partial_\rho \phi - \nabla_\rho \partial_\sigma \phi = \left(\Gamma^\kappa{}_{\sigma\rho} - \Gamma^\kappa{}_{\rho\sigma} \right) \partial_\kappa \phi \, , \tag{5.17}$$

vanishes for a symmetric connection $\Gamma^\kappa{}_{\rho\sigma} = \Gamma^\kappa{}_{\sigma\rho}$, like the metric connection we are working with (cf. Eq. (4.62)). Taking this into account, let me compute the quantity

$$\left[\nabla_\sigma, \nabla_\rho \right] \left(\phi u^\mu \right) = \left(\left[\nabla_\sigma, \nabla_\rho \right] \phi \right) u^\mu + \phi \left[\nabla_\sigma, \nabla_\rho \right] u^\mu = \phi \left[\nabla_\sigma, \nabla_\rho \right] u^\mu \, . \tag{5.18}$$

The final result has important consequences. In particular, it tells us that $\left[\nabla_\sigma, \nabla_\rho \right] u^\mu$ cannot depend on the derivatives of u^ρ because in that case it would also have to depend on the derivatives of the scalar field ϕ. As the dependence on the vector u^μ is linear, we are left with an expression of the form

$$\left[\nabla_\sigma, \nabla_\rho \right] u^\mu = R^\mu{}_{\nu\sigma\rho} u^\nu \, , \tag{5.19}$$

with $R^\mu{}_{\nu\rho\sigma}$ some unknown coefficients. Although the particular combination of connections inside these coefficients cannot be determined without performing the full computation, it is nice to have an idea of the final result before computing it, right?

Let us start the explicit computation of the commutator $\left[\nabla_\sigma, \nabla_\rho \right] u^\mu$ from the definition of the covariant derivative

$$\nabla_\rho u^\mu = \partial_\rho u^\mu + \Gamma^\mu{}_{\kappa\rho} u^\kappa \, . \tag{5.20}$$

Differentiating with respect to x^σ we obtain

$$\begin{aligned} \nabla_\sigma \nabla_\rho u^\mu &= \partial_\sigma \left(\nabla_\rho u^\mu \right) + \Gamma^\mu{}_{\lambda\sigma} \nabla_\rho u^\lambda - \Gamma^\kappa{}_{\rho\sigma} \nabla_\kappa u^\mu \\ &= \partial_\sigma \partial_\rho u^\mu + \partial_\sigma \left(\Gamma^\mu{}_{\kappa\rho} u^\kappa \right) + \Gamma^\mu{}_{\lambda\sigma} \left(\partial_\rho u^\lambda + \Gamma^\lambda{}_{\kappa\rho} u^\kappa \right) - \Gamma^\kappa{}_{\rho\sigma} \left(\partial_\kappa u^\mu + \Gamma^\mu{}_{\lambda\kappa} u^\lambda \right) \, , \end{aligned} \tag{5.21}$$

where we have treated $\nabla_\rho u^\mu$ as a second rank tensor. Computing the difference $\nabla_\sigma \nabla_\rho u^\mu - \nabla_\rho \nabla_\sigma u^\mu$ we find that the terms involving first derivatives of u^μ vanish, as expected. The covariant derivatives of vectors do not commute by a value that depends only on the vector field at the point in question

$$[\nabla_\sigma, \nabla_\rho]\, u^\mu = -\left(\partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho}\right) u^\nu \equiv -R^\mu{}_{\nu\rho\sigma} u^\nu = R^\mu{}_{\nu\sigma\rho} u^\nu \,. \tag{5.22}$$

Ambiguities

Note that the non-commutation of covariant derivatives gives rise to some ambiguities in the *minimal coupling* prescription (*colon-goes-to-semicolon*) introduced in the previous Chapter. To illustrate this, consider for instance a physical law which in an inertial frame takes the form

$$U^\mu \partial_\mu \partial_\nu V^\nu = U^\mu \partial_\nu \partial_\mu V^\nu = 0 \,, \tag{5.23}$$

with U^μ and V^ν some vector fields. Which should be the covariant generalization of this law? Should we write something like

$$U^\mu \nabla_\mu \nabla_\nu V^\nu = 0 \,, \tag{5.24}$$

or rather something like

$$U^\mu \nabla_\nu \nabla_\mu V^\nu = 0\,? \tag{5.25}$$

According to (5.22), these two equations are not equal; they differ by a factor proportional $R^\mu{}_{\nu\rho\sigma}$, which is not necessarily zero. The *colon-goes-to-semicolon* prescription is ambiguous. This is reminiscent of the problem of ordering operators in quantum mechanics: the minimal prescription does not say anything about how to order the operators. The correct way of adapting the laws of physics to spaces with non-vanishing $R^\mu{}_{\nu\rho\sigma}$ can be only determined by experiments.

The n^4 quantities

$$R^\mu{}_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu{}_{\nu\sigma} - \partial_\sigma \Gamma^\mu{}_{\nu\rho} + \Gamma^\mu{}_{\kappa\rho}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\rho} \tag{5.26}$$

are the components of a tensor, as can be easily seen by applying the quotient theorem[2] to Eq. (5.22). This tensor is called the *curvature or Riemann tensor* and it is defined in terms of the metric and its first and second derivatives.

Exercise:

- Which is the value of $R^\mu{}_{\nu\rho\sigma}$ for a 2 dimensional Euclidean metric written in Cartesian coordinates? And if the metric is written in polar coordinates?
- Derive the action of the commutator of two covariant derivatives on a covariant vector. Hint: This should be a fast exercise. Remember the metric compatibility.
- Use the previous result to determine the action of the commutator of covariant derivatives on an arbitrary rank-(r, s) tensor.

Substituting (5.22) into Eq. (5.16) we obtain the so-called *geodesic deviation* equation

$$\frac{D^2 v^\mu}{d\sigma^2} = -R^\mu{}_{\nu\rho\sigma} u^\nu u^\sigma v^\rho \,. \tag{5.27}$$

The term in the right-hand side is the sought-after effect of gravity that cannot be removed by going to a free falling frame: the tidal acceleration. In the non-relativistic limit, the intrinsic derivative on

[2] cf. property 3 in Section 1.4.4

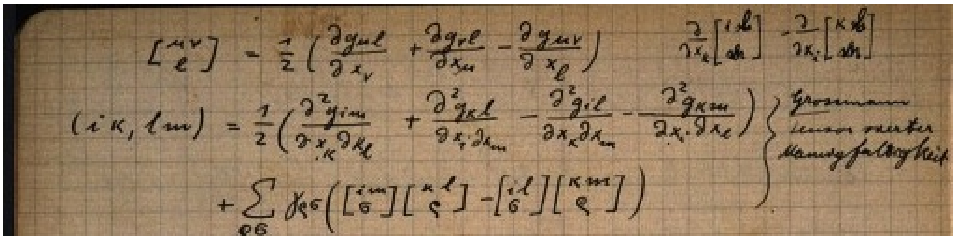

Figure 5.3: First appearance of the Riemann tensor in Einstein's Zurich notebooks. The Riemann tensor is written in the old-fashioned notation (ik, lm). According to some urban legends, Einstein learned the methods of Ricci and Levi-Civita through his school friend Marcel Grossmann. It was Grossmann the one who went to the library searching for methods to deal with arbitrary coordinate systems and discovered the Ricci and Levi-Civita's 1901 paper. The annotation "Grossmann tensor fourth rank" that you can find in the right hand side of the formula suggests indeed that Grossmann conveyed the Riemann tensor formula to Einstein.

the left hand side becomes d^2/dt^2 and $u^\mu \approx \delta^\mu{}_0$, in such a way that

$$\frac{d^2 v^\mu}{dt^2} = -R^\mu{}_{0\rho 0} v^\rho \,. \tag{5.28}$$

Taking into account Eqs. (5.4) and (5.5) we can to identify $R^\mu{}_{0\rho 0}$ with the non-relativistic tidal tensor [3]

$$E^i{}_j = R^i{}_{0j0} \,. \tag{5.30}$$

Exercise:
Compute the Christoffel symbols and the curvature tensor to the lowest order for the line element

$$ds^2 = -(1+2\phi)\, dt^2 + \delta_{ij} dx^i dx^j . \tag{5.31}$$

Interpret the result.

5.3 Flat versus curved: A dirty and quick introduction to curvature.

The geodesic equation is a clear manifestation of the geometrical character of the Einstein's theory of gravity: it is a theory of curved spacetimes. To understand this, let me start with a basic and dirty introduction to the theory of surfaces and the concept of *curvature*. When I say curvature I mean what you understand by curvature in your everyday experience; objects such as eggshells, donuts, tennis balls, etc... are curved. A two dimensional surface can be though as embedded in the usual

[3] Note that the deviation between two neighboring geodesics parametrized by the values λ and $\lambda + d\lambda$ is given by

$$\xi^\mu = \frac{dx^\mu}{\partial \lambda} \delta\lambda = v^\mu \delta\lambda \,. \tag{5.29}$$

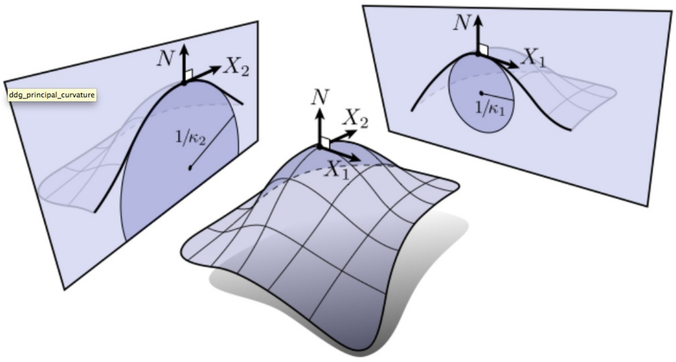

Figure 5.4: Principals curvatures of a surface.

3-dimensional Euclidean space[4]. At any given point P on the 2-dimensional surface, we can introduce a tangent plane with Cartesian coordinates (X_1, X_2) (cf. Fig. 5.4). This Euclidean space is called the *tangent space* to the surface at P. The deviation $z(X_1, X_2)$ of the curved surface from the tangent plane describes the local properties of our geometry. Since curvature effects arise only through the second derivatives of $z(x, y)$, it is convenient to use a quadratic function

$$z(X_1, X_2) = \frac{1}{2}\mathbf{X}^T M \mathbf{X}\,, \tag{5.32}$$

with

$$M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \qquad \mathbf{X} \equiv (X_1, X_2)^T \ , \tag{5.33}$$

and a, b and c quantities with dimensions of inverse length[5]. Eq. (5.32) can be recast in a diagonal form by rotating the coordinates, $\bar{\mathbf{X}} = R\mathbf{X}$, and accordingly transforming the matrix M, $\bar{M} = R^{-1}MR$. In the new coordinate basis (ξ, η),we obtain

$$z(\xi, \eta) = \frac{1}{2}\left(\kappa_1 \xi^2 + \kappa_2 \eta^2\right) \equiv \frac{1}{2}\left(\frac{\xi^2}{\rho_1} + \frac{\eta^2}{\rho_2}\right) , \tag{5.34}$$

where we have defined the so-called *principal curvatures* κ_1 and κ_2 and the *principal radii* of curvature ρ_1 and ρ_2.

The result is quite intuitive. It simply states that any surface is locally the sum of two parabolas in the ξ and η directions and with radius of curvature ρ_1 and ρ_2 respectively (cf. Fig. 5.4).

[4] We do this just for visualization purposes; that is why I said that my introduction is somehow dirty. There is no need to choose a particular embedding for studying the geometry of the surface; the geometry can be completely determined by measuring angles and distances on the surface. This is indeed a theorem, known as *Gauss' Egregium Theorem*. It words of Gauss himself, it reads

> *Formula itaque art[iculi] praec[edentis] sponte perducit ad egregium Theorema. Si superficies curva in quamcunque aliam superficiem explicatur, mensura curvaturae in singulis punctis invariata manet,*

which, for those of you not knowing latin means

> *Thus the formula of the preceding article leads itself to the remarkable Theorem. If a curved surface is developed upon any other surface whatever, the measure of curvature in each point remains unchanged.*

[5] A *local* region is defined for values of X_1 and X_2 much smaller than a^{-1}, b^{-1}, c^{-1}

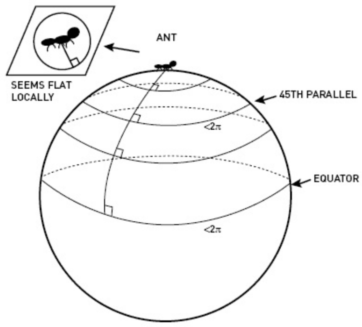

Figure 5.5: A clever ant determining the curvature of a sphere via the Bertrand-Diquet-Puiseux formula.

Exercise
Expand a circle of radius ρ around some point. Comment on the result.

The square of the distance between two nearby points with coordinates[6] (x, y) and $(x + dx, y + dy)$ is given by

$$ds^2 = d\xi^2 + d\eta^2 + dz^2 = (\kappa_1 \xi d\xi + \kappa_2 \eta d\eta)^2 + \left(d\xi^2 + d\eta^2\right) \equiv \gamma_{\mu\nu} dx^\mu dx^\nu \,. \tag{5.35}$$

Since the measure of the surface curvature cannot depend on the set of coordinates used, it must be related to the basis-independent attributes of the matrix M. These attributes are its eigenvalues, or equivalently, its determinant and trace. The determinant $K = \det M = \kappa_1\kappa_2$ is called *intrinsic or Gaussian curvature* and can be expressed entirely in terms of intrinsic measurements on the surface, without any reference to the external embedding space. Starting from a point P on the surface and proceeding along a geodesic on the surface for a proper distance ϵ, we arrive to a point Q_1. Repeating this process with geodesics starting off in different directions, we obtain a set of points $Q_1, Q_2, \ldots$, all of them sitting at the circumference $C(\epsilon)$ of a geodesic disc centered at P (cf. Fig. 8.6). A simple computation using the metric (5.35) shows that the quantity[7]

$$\lim_{\epsilon\to 0^+} \frac{3}{\pi\epsilon^3} \left(2\pi\epsilon - C(\epsilon)\right) \,, \tag{5.37}$$

measuring the difference between the circumference $C(\epsilon)$ of our geodesic disc and a circumference in the plane, corresponds precisely to the value of the Gaussian curvature K at P

$$K = \kappa_1\kappa_2 = \frac{1}{\rho_1\rho_2} = \lim_{\epsilon\to 0^+} \frac{3}{\pi\epsilon^3} \left(2\pi\epsilon - C(\epsilon)\right) \,. \tag{5.38}$$

This expresion, relating the Gaussian curvature of a surface to the circumference of a geodesic circle, is known as the *Bertrand-Diquet-Puiseux formula*, and is closely related to the Gauss-Bonnet theorem that we will discuss below. Spaces with $K = 0$ everywhere are said to be *flat or developable*, since they can be "developed" or flattened out into a plane without stretching or tearing them (cf. Fig.

[6] Note that although M is diagonal, the metric is not.

[7] There is not an absolute scale for Gaussian curvature, neither a unique choice of the normalization factor $3/\pi\epsilon^3$ appearing in Eq. (5.37). People have just agreed on the convention that the curvature of the unit sphere should be equal to 1 (although there are some natural motivations for it). For a small geodesic disc on the unit sphere of radius ϵ we have

$$C(\epsilon) \sim 2\pi\left(\epsilon - \frac{1}{6}\epsilon^3\right) \,, \tag{5.36}$$

which explains the proportionality factor $3/\pi\epsilon^3$.

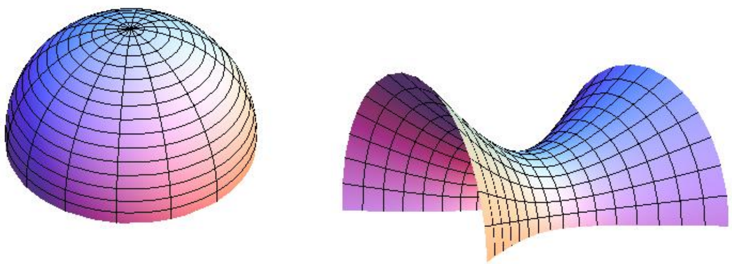

Figure 5.6: Positive ($K > 0$) and negatively curved ($K < 0$) spaces.

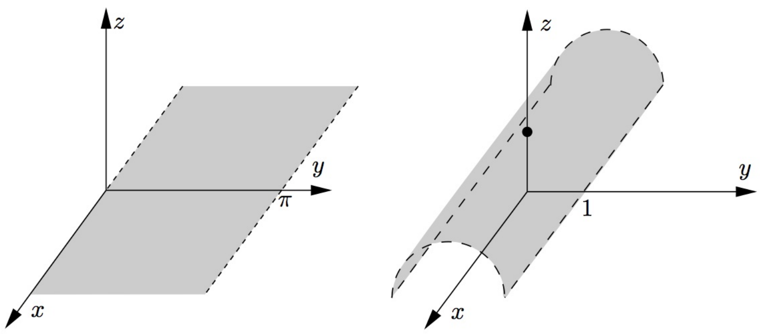

Figure 5.7: A plane sheet of paper ($\kappa_1 = \kappa_2 = 0$) rolled in the form of a cylinder of radius r ($\kappa_1 = 1/r$ and $\kappa_2 = 0$). The extrinsic curvature *changes* from 0 to $\kappa_1 + \kappa_2 = 1/r$.

5.7). On the other hand, spaces with $K > 0$ everywhere are said to be *positively curved*, while spaces with $K < 0$ everywhere are said to be *negatively curved or saddle like.* For someone living on a given point of a space embedded in a higher dimensional space, the curvature at that point will be positive if the space curves away in the same way in any direction, while it will be negative if the space curves away in a different way when moving in different directions (cf. Fig. 5.6).

A worked-out example: a *truly* curved space.

As a direct application of the Bertrand-Diquet-Puiseux formula, consider the metric of the 2-dimensional sphere of unit radius

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \,, \tag{5.39}$$

and take P to be the origin. The distance from the origin to the point (ϵ, θ) is given by

$$\int_0^\epsilon ds = \epsilon. \tag{5.40}$$

The set of points with coordinates (ϵ, θ) form a disc whose circumference is given by

$$\int d\theta \sin\epsilon = 2\pi \sin\epsilon \,. \tag{5.41}$$

Applying (5.38), we get

$$\lim_{\epsilon\to 0} \frac{6}{\epsilon^2}\left(1 - \frac{\sin\epsilon}{\epsilon}\right) = 1 \,. \tag{5.42}$$

The sphere (5.39) is a positively curved space.

On the other hand, the *extrinsic curvature*[8] is defined through the trace of M, namely $\kappa_1 + \kappa_2$. The difference between the two can be easily understood by considering, for instance, a plane sheet of paper ($\kappa_1 = \kappa_2 = 0$) rolled in the form of a cylinder of radius r which will look like a curved 2-dimensional surface embedded in a 3-dimensional Euclidean space (cf. Fig 5.7). For the cylindrical surface we have $\kappa_1 = 1/r$ and $\kappa_2 = 0$. The intrinsic curvature retains the value of the flat sheet of paper. On the other hand, the extrinsic curvature changes from 0 to $\kappa_1 + \kappa_2 = 1/r$.

Exercise: Coodinates should not be trusted

Is the 2-dimensional space $ds^2 = \cos^2\phi d\phi^2 + \sin^2\phi d\theta^2$ curved or flat?

5.4 Parallel transport around a closed path

Consider the sum of the angles of a triangle, let's call them α, β and γ. As you know this sum is equal to π rad in flat space. What happens in a curved surface? When a surface is curved the sum of the angles in the triangle[9] is in general different from π. The more curved the surface is, the larger is the difference with respect to the flat result. The quantified version of this rather intuitive result is

[8] In some books, the extrinsic curvature is normalized as $(\kappa_1 + \kappa_2)/2$ and called *mean curvature*.

[9] We are implicitly assuming that the sides of the triangle are geodesics, the curved analog of Euclidean straight lines.

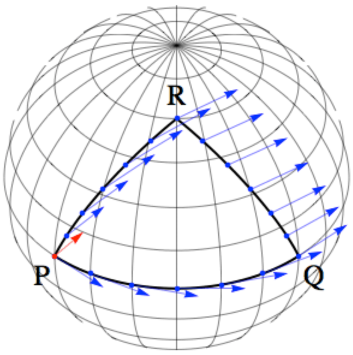

Figure 5.8: Parallel transport of a vector around a closed path on the sphere.

the result of so-called *Gauss-Bonnet theorem*[10]:

$$\int_S K dS = \alpha + \beta + \gamma - \pi \tag{5.43}$$

with K the Gauss curvature and S the area inside the triangle. To generalize this form of curvature, note that when the tangent vector at the PQ side is parallel transported from P to Q (cf. Fig. 5.8), it forms an angle $\pi - \beta$ with the tangent vector of the next side of the triangle. The same happens in the other vertices. This means that if we make a parallel transport around the whole close path, we obtain an angle $\pi - \beta + \pi - \gamma + \pi - \alpha$, which, forgetting about 2π multiples and writing the appropriate sign is given by $\alpha + \beta + \gamma - \pi$. The Gauss curvature measures the variation, in relation with the area, of parallel transported vectors around closed paths.

Ways of determining curvature

- Make distance measurements in different directions to construct the metric and then use it to find the curvature
- Take a vector and go around two different paths.

Note that in both cases, we don't make any reference to the higher-dimensional space in which we are embedded.

Although the intuitive reasoning presented above was bidimensional, it can be easily generalized to arbitrary dimension. To do that consider the parallel transport equation

$$\frac{dv^\mu}{d\sigma} = -\Gamma^\mu{}_{\nu\rho} v^\nu \frac{dx^\rho}{d\sigma} \tag{5.44}$$

and apply it to the case in which v^μ is parallel-transported along a small curve $\mathcal{C}$ from some initial point P. The value of the vector at any other point σ along this curve is given by

$$v^\mu(\sigma) = v^\mu_P - \int_o^\sigma \Gamma^\mu{}_{\nu\rho} v^\nu \frac{dx^\rho}{d\sigma} d\sigma \,. \tag{5.45}$$

Let us assume the loop $\mathcal{C}$ to be infinitesimally small. In that case, the quantities in the integrand of

[10]The standard presentation of the theory of surfaces is usually based on Gauss' Egregium Theorem and finishes with the derivation of the Gauss-Bonnet theorem. This sequence is however not chronological. Gauss deduced the Egregium Theorem starting from the Gauss-Bonnet theorem.

previous expression can be Taylor expanded around the point P to get

$$\Gamma^{\mu}{}_{\nu\rho}(\sigma) = \Gamma^{\mu}{}_{\nu\rho}|_P + \partial_\lambda \Gamma^{\mu}{}_{\nu\rho}\Big|_P \Delta x^\lambda + \dots \quad (5.46)$$

$$v^\mu(\sigma) = v^\mu_P - \Gamma^{\mu}{}_{\nu\rho}\Big|_P v^\nu_P \Delta x^\rho + \dots \quad (5.47)$$

with $\Delta x^\lambda \equiv x^\lambda(\sigma) - x^\lambda_P$. Plugging back these expressions into (5.45) and retaining only those terms up to first order in Δx^λ, we obtain

$$v^\mu(\sigma) = v^\mu_P - \Gamma^{\mu}{}_{\nu\rho}\Big|_P v^\nu_P \int_0^\sigma \frac{dx^\rho}{d\sigma} d\sigma - \left(\partial_\lambda \Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\lambda}\right)\Big|_P v^\nu_P \int_0^\sigma \left(x^\lambda - x^\lambda_P\right) \frac{dx^\rho}{d\sigma} d\sigma \,. \quad (5.48)$$

The second and the last term (the part associated to x^λ_P) vanish for a closed path ($\oint dx^\rho = 0$) . We are left therefore with a net change

$$\Delta v^\mu = -\left(\partial_\lambda \Gamma^{\mu}{}_{\nu\rho} - \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\lambda}\right)\Big|_P v^\nu_P \oint x^\lambda dx^\rho . \quad (5.49)$$

This effect can be written in a more meaningful form by adding the result of interchanging the dummy indices ρ and λ. Doing this, and taking into account that

$$\oint d(x^\rho x^\lambda) = \oint \left(x^\rho dx^\lambda + x^\lambda dx^\rho\right) = 0 \,, \quad (5.50)$$

we get

$$\Delta v^\mu = -\frac{1}{2}\left(\partial_\rho \Gamma^{\mu}{}_{\nu\lambda} - \partial_\lambda \Gamma^{\mu}{}_{\nu\rho} + \Gamma^{\mu}{}_{\kappa\rho}\Gamma^{\kappa}{}_{\nu\lambda} - \Gamma^{\mu}{}_{\kappa\lambda}\Gamma^{\kappa}{}_{\nu\rho}\right)\Big|_P v^\nu_P \oint x^\rho dx^\lambda \,. \quad (5.51)$$

Denoting by

$$A^{\rho\lambda} \equiv \oint x^\rho dx^\lambda \quad (5.52)$$

the total area enclosed by the loop $\mathcal{C}$ and taking into account Eq. (5.26), we finally obtain

$$\Delta v^\mu = -\frac{1}{2} R^\mu{}_{\nu\rho\lambda} v^\nu_P A^{\sigma\lambda} \,. \quad (5.53)$$

The change of the vector when it moves along a closed path is proportional to the Riemann tensor and to the area enclosed by the loop[11]! $R^\mu{}_{\nu\rho\sigma}$ is the generalization[12] of the Gauss curvature K. The components of a vector v^μ will remain unchanged after parallel transport *if and only if* the curvature tensor vanishes. In that happens, the spacetime is actually flat. Any apparent dependence of the metric on the coordinates will be just an illusion due to the use of some weird coordinate system and

[11] Note that although our derivation was performed under the assumption of having an infinitesimal loop, it can be easily extended to larger closed curves. A given surface A bounded by a curve $\mathcal{C}$ can be understood as the sum of many small areas bounded by closed curves $\mathcal{C}_N$. Since the changes in Δv^μ around any of the interior curves cancel and only the outer edges contribute, we can express the change in the components v^μ along $\mathcal{C}$ as the sum of the changes around the small curves, namely

$$\Delta v^\mu = \sum_N (\Delta v^\mu)_N \,. \quad (5.54)$$

[12] Indeed the geodesic deviation equation (5.27) is nothing else than the generalization of the Jacobi equation

$$\frac{d^2 y}{d\sigma^2} + K y = 0 \quad (5.55)$$

between two geodesics in a two dimensional surface.

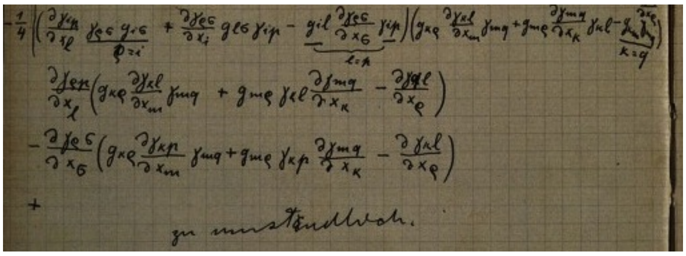

Figure 5.9: Einstein's manipulations of the Riemann tensor (Zurich notebook). The computation is abandoned, "zu umstaendlich" (too involved).

we will be able to find a *global* coordinate system in which the metric takes a Cartesian form.

> **Exercise:**
> Determine the Gauss curvature of a spherical surface of radius R through the Gauss-Bonnet theorem. Hint: Apply it, for instance, to the triangle determine by the 1/8 part of the sphere.

5.5 Properties of the Riemann tensor

Eq. (5.26) provides a way of computing the 256 components of the Riemann tensor directly from the line element. This is usually a rather tedious process, even for Einstein (cf. Fig. 5.9). Fortunately, the covariant form of the Riemann tensor $R_{\mu\nu\rho\sigma} \equiv g_{\mu\lambda} R^{\lambda}{}_{\nu\rho\sigma}$ shows many interesting symmetries in its indices that will simplify our life. Writing it explicitly in terms of the metric and Christoffel symbols we get

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu\partial_\rho g_{\mu\sigma} + \partial_\mu\partial_\sigma g_{\nu\rho} - \partial_\nu\partial_\sigma g_{\mu\rho} - \partial_\mu\partial_\rho g_{\nu\sigma}\right) + g_{\lambda\kappa}\left(\Gamma^{\lambda}{}_{\nu\rho}\Gamma^{\kappa}{}_{\mu\sigma} - \Gamma^{\lambda}{}_{\nu\sigma}\Gamma^{\kappa}{}_{\mu\rho}\right) . \tag{5.56}$$

Using this expression we can derive the following properties:

- **Symmetry**: The Riemann tensor $R_{\rho\sigma\mu\nu}$ is symmetric under the interchange of the first *pair* of indices with the second *pair* of indices

$$R_{\mu\nu\rho\sigma} = +R_{\rho\sigma\mu\nu} . \tag{5.57}$$

- **Antisymmetry**: The Riemann tensor $R_{\rho\sigma\mu\nu}$ is antisymmetric under the interchange of either the first two indices or the second two indices

$$R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\nu\mu\sigma\rho} . \tag{5.58}$$

  This is a direct consequence of the definition of the Riemann tensor (the operator $[\nabla_\sigma, \nabla_\rho]$ is antisymmetric) and the metric compatibility

$$[\nabla_\sigma, \nabla_\rho] g_{\mu\nu} = 0 \qquad \longrightarrow \qquad R^{\kappa}{}_{\mu\rho\sigma} g_{\kappa\nu} + R^{\kappa}{}_{\nu\rho\sigma} g_{\mu\kappa} = (R_{\nu\mu\sigma\rho} + R_{\mu\nu\rho\sigma}) = 0 . \tag{5.59}$$

- **1st Bianchi identity**: The cyclic sum of the last three indices is zero

$$3R_{\mu[\nu\rho\sigma]} \equiv R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0 . \tag{5.60}$$

This can be easily understood by applying the operator $[\nabla_\rho, \nabla_\sigma]$ to the gradient $\nabla_\nu \phi$ of a scalar field. For any scalar $\nabla_{[\rho}\nabla_\sigma\nabla_{\nu]}\phi = 0$, which implies

$$R^\kappa{}_{[\nu\rho\sigma]}\nabla_\kappa\phi = 0\,. \tag{5.61}$$

Since the resulting expression is valid for all gradients, Eq. (5.60) follows immediately. Note that the result is non-trivial only when the three indices $\nu\rho\sigma$ are different. When two of these indices are equal one of the terms drop and the remaining terms just express the antisymmetry in the last two indices of the curvature tensor.

- **2nd Bianchi identity**: The Riemann tensor satisfies the differential identity[13]

$$\nabla_\kappa R^\mu{}_{\nu\rho\sigma} + \nabla_\sigma R^\mu{}_{\nu\kappa\rho} + \nabla_\rho R^\mu{}_{\nu\sigma\kappa} = 0\,. \tag{5.63}$$

The proof is left as an exercise.

Exercise

Prove Eq. (5.63) Hint: Use a local inertial frame.

- **Ricci tensor and Ricci scalar**: There are two important contractions of the Riemann tensor[14]. The first one is a second rank tensor obtained from contracting a pair of indices. Since $R_{\mu\nu\rho\sigma}$ is antisymmetric in $\mu\nu$ and $\rho\sigma$, the only non-trivial contraction is between μ and ρ or between μ and σ. These two contractions differ only by a change of sign. Taking the first contraction, we obtain the so-called *Ricci tensor*

$$R_{\nu\sigma} \equiv g^{\mu\rho}R_{\mu\nu\rho\sigma} = R^\mu{}_{\nu\mu\sigma} = \partial_\mu\Gamma^\mu{}_{\nu\sigma} - \partial_\sigma\Gamma^\mu{}_{\nu\mu} + \Gamma^\mu{}_{\kappa\mu}\Gamma^\kappa{}_{\nu\sigma} - \Gamma^\mu{}_{\kappa\sigma}\Gamma^\kappa{}_{\nu\mu}\,, \tag{5.64}$$

which is symmetric, as can be easily seen by taking into account the relation (4.72)

$$\partial_\sigma\Gamma^\mu{}_{\nu\mu} = \partial_\sigma\left(\frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|}\right) = -\frac{1}{|g|}\partial_\sigma\sqrt{|g|}\partial_\nu\sqrt{|g|} + \frac{1}{\sqrt{|g|}}\partial_\nu\partial_\sigma\sqrt{|g|}\,. \tag{5.65}$$

The second contraction is the so-called *Ricci scalar or Ricci curvature*

$$R \equiv R^\nu{}_\nu = g^{\nu\sigma}R_{\nu\sigma} = g^{\mu\rho}g^{\nu\sigma}R_{\mu\nu\rho\sigma}\,. \tag{5.66}$$

That's all. There are no more non-vanishing contractions. The result (5.66) is quite remarkable. Among the 20 independent components of the Riemann tensor that transform into linear combinations of each other under general coordinate transformations, there is *one* which remains unchanged. R is the only scalar involving the metric and two derivatives.

Exercise:

Among the different ways of constructing a scalar from the Riemann tensor discussed above, why did I not discuss the contraction $\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$?

[13] This identity is related to the Jacobi identity

$$[[\nabla_\mu, \nabla_\nu], \nabla_\rho] + [[\nabla_\nu, \nabla_\rho], \nabla_\mu] + [[\nabla_\rho, \nabla_\mu], \nabla_\nu] = 0\,. \tag{5.62}$$

[14] We will only discuss the contractions at the lower order in the curvature tensor. Higher order contractions such as R^2, $R_{\mu\nu}R^{\mu\nu}$ or the square of the Riemann tensor, the so-called *Kretschmann* scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, will be introduced at its due time.

- **Contracted Bianchi identities**: Note the important result that follows from the Bianchi identity (5.63) and the definition of the Ricci scalar. Contracting the indices $\mu\rho$ in (5.63) we get

$$\nabla_\kappa R^\rho{}_{\nu\rho\sigma} + \nabla_\sigma R^\rho{}_{\nu\kappa\rho} + \nabla_\rho R^\rho{}_{\nu\sigma\kappa} = \nabla_\kappa R_{\nu\sigma} - \nabla_\sigma R_{\nu\kappa} + \nabla_\rho R^\rho{}_{\nu\sigma\kappa} = 0\,, \tag{5.67}$$

where we have made use of the antisymmetry property (5.58). Multiplying by the metric $g^{\nu\sigma}$, contracting the indices ν and σ and taking into account that $\nabla_\rho R^{\rho\sigma}{}_{\sigma\kappa} = -\nabla_\rho R^{\sigma\rho}{}_{\sigma\kappa} = -\nabla_\rho R^\rho{}_\kappa$, Eq. (5.67) becomes

$$\nabla_\kappa R - \nabla_\sigma R^\sigma{}_\kappa - \nabla_\rho R^\rho{}_\kappa \quad = \quad 0\,. \tag{5.68}$$

The previous expression can be written in a much more enlightening way

$$\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0\,. \tag{5.69}$$

The divergence of the so-called *Einstein tensor*

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{5.70}$$

vanishes by construction[15]! The symmetry of the Einstein tensor under the interchange of its indices follows directly from the symmetries of the Ricci tensor and the metric. Which is the geometrical meaning of this tensor? To answer this, consider an observer moving with 4-velocity u^μ and compute the spatial components of the Riemann tensor in the instantaneous rest frame of such an observer[16]

$$\mathcal{R}_{\gamma\epsilon\lambda\kappa} = h^\mu{}_\gamma h^\nu{}_\epsilon h^\rho{}_\lambda h^\sigma{}_\kappa R_{\mu\nu\rho\sigma} \tag{5.71}$$

where we have made used of the projection operator $h^\mu{}_\nu = \delta^\mu{}_\nu + u^\mu u_\nu$. Contracting the indices γ and λ and the indices ϵ and κ in the previous expression, we get the scalar

$$\mathcal{R} = h^{\mu\rho} h^{\nu\sigma} R_{\mu\nu\rho\sigma} = \left(g^{\mu\rho} + u^\mu u^\rho\right)\left(g^{\nu\sigma} + u^\nu u^\sigma\right) R_{\mu\nu\rho\sigma} = R + 2u^\mu u^\rho R_{\mu\rho}\,. \tag{5.72}$$

which, comparing with the definition (5.70) of the Einstein tensor , can be written as

$$\mathcal{R} = 2u^\mu u^\rho G_{\mu\rho}\,. \tag{5.73}$$

$G_{\mu\nu} u^\mu u^\nu$ measures the local scalar curvature of the spatially projected curvature tensor.

A final warning

There are several sign conventions involved in the definition of the Riemann tensor and its contractions. Be careful when taking results from different books or articles. Our convention is that of Misner, Thorne and Wheeler. A very useful reference sheet taken precisely from this book can be found in the Moodle.

5.6 Independent components of the Riemann tensor

How many independent components has the Riemann tensor $R_{\mu\nu\rho\sigma}$ in n dimensions? As a 4-indexed object in n dimensions we have a priori n^4 independent components, but the symmetries (5.57)-(5.60) will significantly reduce this number. In order to see this, consider the Riemman tensor $R_{\mu\nu\rho\sigma}$ as

[15] Remember this, we will made use of it very soon.
[16] $\mathcal{R}_{\mu\nu\rho\sigma}$ is *not* the curvature of the 3-space orthogonal to u^μ, ${}^{(3)}R_{\mu\nu\rho\sigma}$!

the expression of a symmetric $m \times m$ matrix[17] $R_{AB} = R_{BA}$ with indices $A = \{\mu\nu\}$ and $B = \{\rho\sigma\}$. This matrix has $\frac{1}{2}m(m+1)$ independent components. The value of m is determined by the number of choices that we have for A and B, which, taking into account Eq.(5.58), have the same content as a $n \times n$ antysimmetric matrix. We have therefore $m = \frac{1}{2}n(n-1)$ possible choices of A and B. The total number of components so far is

$$\frac{m(m+1)}{2} = \frac{1}{2}\left(\frac{n(n-1)}{2}\right)\left(\frac{n\,(n-1)}{2}+1\right) = \frac{\left(n^4 - 2n^3 + 3n^2 - 2n\right)}{8}, \tag{5.74}$$

but we have still to substract the constraints imposed by Eq.(5.60). To determine the number of extra constraints, notice that if one sets any two components equal (for instance $\mu = \nu$) we get identically zero (one term goes away by antisymmetry and the other two cancel). Only if the 4 indices are different we get a constraint. The number of independent constraints is the same as the number of combinations of 4 objects that can be chosen from n objects

$$\binom{n}{4} = \frac{n!}{4!(n-4)!} = \frac{n(n-1)(n-2)(n-3)}{24}. \tag{5.75}$$

The final number of independent components of the Riemann tensor becomes

$$C_R = \frac{m(m+1)}{2} - \frac{n!}{4!(n-4)!} = \frac{n^2(n^2-1)}{12}. \tag{5.76}$$

Evaluating this for different dimensions we get

Number of dimensions	1	2	3	4	5
Total components of $R^{\mu}{}_{\nu\rho\sigma}$	1	16	81	256	625
Independent components of $R^{\mu}{}_{\nu\rho\sigma}$	0	1	6	20	50

The number of independent components in 4 dimensions has been reduced from 256 to 20! The fact that the number is still quite large is reasonable, since we need a lot of numbers to specify how the space curves in many different directions.. As we will see in the next Section, these are precisely the degrees of freedom in the second derivatives of the metric that we cannot set to zero by performing a change of coordinates.

Exercise

- In one dimension the Riemann tensor is always identically zero. Explain why.

  Hint: Remember the geometrical interpretation of the Riemann tensor.

- How many components have the Ricci tensor and the Ricci scalar in 2, 3 and 4 dimensions? And the Einstein tensor? Is there any dimension in which the Riemann and the Ricci tensors haves the same number of independent components?

5.6.1 Local versus global flatness: A counting exercise

The Equivalence Principle is based on the existence of locally inertial (or freely falling) reference frames

$$g_{\mu\nu}(P) = \eta_{\mu\nu}\,, \qquad \partial_\sigma g_{\mu\nu}(P) = 0\,, \tag{5.77}$$

[17] This is sometimes called the *Petrov notation*.

in which gravity can be transformed away. So, one of the things that we will like to verify is that this kind of coordinate systems exist in the context of Riemannian geometry, i.e., if we can always introduce a free falling frame (5.77) at an arbitrary point for an arbitrary metric $g_{\mu\nu}$. For doing that, consider a coordinate transformation from the coordinates x^μ to some coordinates ξ^α in the neighborhood of some point P. Performing a Taylor expansion around P, we get

$$\xi^\alpha(x) = \xi^\alpha(P) + A^\alpha{}_\mu \Delta x^\mu + B^\alpha_{\mu\nu}\Delta x^\mu \Delta x^\nu + C^\alpha{}_{\mu\nu\rho}\Delta x^\mu \Delta x^\nu \Delta x^\rho + \dots\,, \tag{5.78}$$

with $\Delta x^\mu \equiv x^\mu - P^\mu$ and[18]

$$A^\alpha{}_\mu = \frac{\partial \xi^\alpha}{\partial x^\mu}\Big|_P\,, \qquad B^\alpha{}_{\mu\nu} = \frac{1}{2}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}\Big|_P\,, \qquad D^\alpha{}_{\mu\nu\rho} = \frac{1}{6}\frac{\partial^3 \xi^\alpha}{\partial x^\mu \partial x^\nu \partial x^\rho}\Big|_P\,. \tag{5.79}$$

Let us see if we can generically choose the values of the coefficients $A^\alpha{}_\mu, B^\alpha{}_{\mu\nu}, D^\alpha_{\mu\nu\rho}\dots$ in such a way that the conditions (5.77) are satisfied[19]. In four dimensions, the matrix $A^\alpha{}_\mu$ has $4^2 = 16$ independent components. Since we need only 10 conditions to impose $g_{\mu\nu}(P) = \eta_{\mu\nu}$, we are left with 6 components to spare, precisely the number of Lorentz transformations and rotations that we can make without modifying the form of metric in the Minkowski metric $\eta_{\mu\nu}$! The requirement $\partial_\sigma g_{\mu\nu}(P) = 0$ give rise to $4 \times 4(4+1)/2 = 40$ conditions, which are precisely the number of components of the symmetric quantity $B^\alpha{}_{\mu\nu}$. We have just proven that one can always choose coordinates in such a way that the metric reduces to the inertial form (5.77) in an *infinitesimal region* around a point P. In the mathematical literature, this is known as the *local flatness theorem*.

But, what happens with the other coefficients? Can we make also put the second derivatives of the metric to zero by simply performing coordinates transformations? The answer is no. The second derivatives of the metric, $\partial_\sigma \partial_\rho g_{\mu\nu}$, have $10 \times 10 = 100$ independent components, while $D^\alpha_{\mu\nu\rho}$ has only $4^2 \times (5 \times 6)/6 = 80$ components. This means that among the 100 components of the metric second derivatives only 80 can be set to zero at P via coordinate transformations. Precisely the number of independent components of the Riemann tensor in 4 dimensions! Indeed, it is not difficult to prove that, at quadratic order in the coordinates, we can write

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}\left(R_{\mu\rho\nu\sigma} + R_{\nu\rho\mu\sigma}\right)\Delta x^\rho \Delta x^\sigma \tag{5.80}$$

The second derivatives of the metric (or if you want the first derivative of the Christoffel symbols) encode the information about the *true* gravitational field $R_{\mu\nu\rho\sigma}$!. A free falling observer can pretend that he/she is not in the presence of a gravitational field, but the tidal forces cannot be eliminated!

Exercise

Repeat this exercise in arbitrary dimensions. What happens?

5.6.2 The Weyl tensor

In 4 dimensions, the Riemann tensor has 20 independent components, while the Ricci tensor and the scalar of curvature can only account for 10+1 of those components. This should be somehow expected, since the Ricci tensor and the scalar curvature contain the information about the "traces" of the Riemann tensor, and not of it as a whole. The 20 independent components of the Riemann curvature tensor in 4 dimensions can be written in terms of three irreducible pieces: the scalar curvature R, the tracefree part of Ricci tensor

$$S_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R\,, \tag{5.81}$$

[18]Note that, in spite of the appearances, the coefficients in the previous expression are not tensors, because they only transform as such under *global linear coordinate transformations.*

[19]Note that the coefficients $B^\alpha{}_{\mu\nu}, C^\alpha_{\mu\nu\rho}\dots$ are completely symmetric in the lower indices.

and the so-called *Weyl tensor*

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{1}{3}Rg_{\mu[\rho}g_{\sigma]\nu}\,. \tag{5.82}$$

The Weyl tensor is a linear rank-(0,4) tensor in $R_{\mu\nu\rho\sigma}$ with no dependence on the derivatives of the metric except through $R_{\mu\nu\rho\sigma}$. It has indeed the same symmetry properties as the Riemann tensor, and therefore the same number of potential components. Note however that the Weyl tensor is traceless

$$C^{\mu}{}_{\nu\mu\sigma} = g^{\mu\rho}C_{\mu\nu\rho\sigma} = 0\,. \tag{5.83}$$

which, taking into account the symmetry in the indices ν and σ, leaves as with $20-10 = 10$ independent components, which together with the $10-1 = 9$ independent components of the trace free part of the Ricci tensor $S_{\mu\nu}$, and the single component of the curvature scalar R, makes the 20 components of the Riemann tensor. Note that no new quantities can be obtained by contracting the indices of the above *irreducible components*.

An important property of the Weyl tensor is its behaviour under *conformal transformations*. A conformal transformation can be understood as a local dilatation, in which the line element changes from ds^2 to $\Omega^2(x)ds^2$, with $\Omega^2(x)$ an arbitrary and non-vanishing function called *conformal factor*[20]. Through a trivial, but quite involved computation, one can verify that when we perform one of these conformal transformations

$$g_{\mu\nu} \longrightarrow \Omega^2(x)g_{\mu\nu}\,, \tag{5.85}$$

the totally covariant Weyl tensor transforms accordingly

$$C_{\mu\nu\rho\sigma} = \Omega^2(x)C_{\mu\nu\rho\sigma}\,, \tag{5.86}$$

and therefore[21] $C^{\mu}{}_{\nu\rho\sigma}$ is conformally invariant[22]. This has an interesting consequence: in those case in which the metric can be written as the result of the conformal transformation of a flat spacetime, $g_{\mu\nu} = f(x)\delta_{\mu\nu}$ or $g_{\mu\nu} = f(x)\eta_{\mu\nu}$, the Weyl tensor is zero and the Riemann tensor can be entirely expressed in terms of the Ricci tensor $R_{\mu\nu}$ and the scalar of curvature R.

Exercise

Prove that the Weyl tensor (5.82) is indeed traceless.

5.7 A laboratory for Riemannian geometry: 2 dimensional manifolds

In two dimensions the covariant Riemann tensor $R_{\mu\nu\rho\sigma}$ has only one independent component. Since the indices can take only two different values, say 1 and 2, and $R_{\mu\nu\rho\sigma}$ is antisymmetric in μ and ν and ρ and σ, and symmetric in the interchange of the combinations $\mu\nu$ and $\rho\sigma$ as a whole, we are left with an expression of the form R_{1212}. Let us see how this component is related to the Ricci scalar. In order to do that, let me express the Riemann tensor as a linear combination of two tensors

$$S_{\mu\nu\rho\sigma} = g_{\mu\rho}g_{\nu\sigma}\,, \qquad T_{\mu\nu\rho\sigma} = g_{\mu\sigma}g_{\nu\rho}\,, \tag{5.87}$$

[20] Note that this kind of transformations conserve the angle between vectors

$$\cos(U,V) = \frac{U_\mu V^\mu}{\sqrt{(U_\nu U^\nu)\,(V_\rho V^\rho)}}\,. \tag{5.84}$$

[21] Note the position of the indices

[22] This is true in any dimension

depending only in the metric and respecting the symmetries of the Riemann tensor[23]

$$R_{\mu\nu\rho\sigma} = A\left(S_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma}\right) . \tag{5.88}$$

Contracting the previous expression to obtain the Ricci scalar in the left-hand side we get

$$R = Ag^{\mu\rho}g^{\nu\sigma}\left(S_{\mu\nu\rho\sigma} - T_{\mu\nu\rho\sigma}\right) = A\left(g^{\mu\rho}g_{\mu\rho}g^{\nu\sigma}g_{\nu\sigma} - g^{\mu\rho}g_{\mu\sigma}g^{\nu\sigma}g_{\nu\rho}\right) = (4-2)A = 2A\,, \tag{5.89}$$

which allows as to identify the unknown factor A in Eq. (5.88) and write the fully covariant expression[24]

$$R_{\mu\nu\rho\sigma} = K\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\mu\rho}\right) , \tag{5.91}$$

where we have defined the Gaussian curvature as $K = R/2$.

5.7.1 A worked-out example: 2 dimensional sphere

Let us go trough the whole process of computing the Ricci scalar. This kind of computations are usually involved, but with a bit of practice and care they are quite tractable[25]. The line element on the surface of a sphere of radius a can be obtained by substituting the coordinate transformations

$$x = a\sin\theta\cos\phi\,, \qquad y = a\sin\theta\sin\phi\,, \qquad z = a\cos\theta\,,$$

into the Euclidean line element $ds^2 = dx^2 + dy^2 + dz^2$. We obtain

$$ds^2 = a^2 d\theta^2 + a^2\sin^2\theta d\phi^2 \qquad \longrightarrow \qquad g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} . \tag{5.92}$$

The Christoffel symbols can be computed in many different ways, being the most practical one the Lagrangian method. The only non-vanishing terms are

$$\Gamma^{\theta}_{\phi\phi} = -\cos\theta\sin\theta\,, \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta\,. \tag{5.93}$$

The $\mu = \theta$ component of the Riemann tensor is given by

$$R^{\theta}{}_{\nu\rho\sigma} = \partial_\rho\Gamma^{\theta}{}_{\nu\sigma} - \partial_\sigma\Gamma^{\theta}{}_{\nu\rho} + \Gamma^{\theta}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\theta}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\nu\rho}\,. \tag{5.94}$$

Among the two possibles values of the indices appearing in the $\Gamma\Gamma$ pieces, only the $\lambda = \rho = \phi$ choice contributes, so we can expand the sum over λ in the last two terms

$$R^{\theta}{}_{\nu\rho\sigma} = \partial_\rho\Gamma^{\theta}{}_{\nu\sigma} - \partial_\sigma\Gamma^{\theta}{}_{\nu\rho} + \Gamma^{\theta}{}_{\phi\rho}\Gamma^{\phi}{}_{\nu\sigma} - \Gamma^{\theta}{}_{\phi\sigma}\Gamma^{\phi}{}_{\nu\rho}\,. \tag{5.95}$$

Since the Riemann tensor is antisymmetric in ρ and σ, we cannot have $\rho = \sigma$. Let's set therefore $\rho = \phi$ and $\sigma = \theta$ (keeping in mind that the alternative choice, $\rho = \theta$ and $\sigma = \phi$, just gives rise to a relative minus sign). We have

$$R^{\theta}{}_{\nu\phi\theta} = \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\nu\theta} - \partial_\theta\Gamma^{\theta}{}_{\nu\phi} = 0\,, \tag{5.96}$$

[23] The combination $S - T$ is antisymmetric under $\mu \leftrightarrow \nu$

[24] This is the particular expression of a much more general relation

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)}\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\mu\rho}\right) . \tag{5.90}$$

for a *maximally symmetric spacetime* with constant R is arbitrary dimension. Unfortunately, I don't have the time to go trough it. The interested reader can have a look to this subject in Weinberg's book.

[25] Since this is the first non-trivial computation of the Ricci scalar that we perform, I will do it in great detail. Although I could directly compute R_{1212} (we are dealing with a 2-dimensional metric) I prefer not to do so in order to teach you some general tricks related to the symmetries of the Riemann tensor that will be useful when dealing with more complicated metrics.

from which we get, potentially, two terms

$$\begin{aligned} R^{\theta}{}_{\theta\phi\theta} &= \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\theta\theta} - \partial_{\theta}\Gamma^{\theta}{}_{\theta\phi} = 0 \qquad (5.97)\\ R^{\theta}{}_{\phi\phi\theta} &= \Gamma^{\theta}{}_{\phi\phi}\Gamma^{\phi}{}_{\phi\theta} - \partial_{\theta}\Gamma^{\theta}{}_{\phi\phi} \\ &= (-\cos\theta\sin\theta)(\cot\theta) - \sin^2\theta + \cos^2\theta = -\sin^2\theta \qquad (5.98)\\ &= -R^{\theta}{}_{\phi\theta\phi}\,. \end{aligned}$$

For $\mu = \phi$, the Riemann tensor becomes

$$R^{\phi}{}_{\nu\rho\sigma} = \partial_{\rho}\Gamma^{\phi}{}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\phi}{}_{\nu\rho} + \Gamma^{\phi}{}_{\lambda\rho}\Gamma^{\lambda}{}_{\nu\sigma} - \Gamma^{\phi}{}_{\lambda\sigma}\Gamma^{\lambda}{}_{\nu\rho}\,. \qquad (5.99)$$

As before, the only option is $\rho = \phi$ and $\sigma = \theta$

$$R^{\phi}{}_{\nu\phi\theta} = \partial_{\phi}\Gamma^{\phi}{}_{\nu\theta} - \partial_{\theta}\Gamma^{\phi}{}_{\nu\phi} + \Gamma^{\phi}{}_{\lambda\phi}\Gamma^{\lambda}{}_{\nu\theta} - \Gamma^{\phi}{}_{\lambda\theta}\Gamma^{\lambda}{}_{\nu\phi}\,. \qquad (5.100)$$

Taking into account that the metric does not depend on ϕ, the previous expression reduces to

$$R^{\phi}{}_{\nu\phi\theta} = -\partial_{\theta}\Gamma^{\theta}{}_{\nu\phi} - \Gamma^{\phi}{}_{\phi\theta}\Gamma^{\phi}{}_{\nu\phi}\,, \qquad (5.101)$$

which is different from zero only if $\nu = \theta$

$$R^{\phi}{}_{\theta\phi\theta} = -\partial_{\theta}\Gamma^{\theta}{}_{\theta\phi} - \Gamma^{\phi}{}_{\phi\theta}\Gamma^{\phi}{}_{\theta\phi} = \frac{1}{\sin^2\theta} - \cot^2\theta = 1\,. \qquad (5.102)$$

The Ricci tensor is obtained by contracting the upper and second lower index. In matrix notation we have

$$R_{\mu\nu} = \begin{pmatrix} R^{\theta}{}_{\theta\theta\theta} + R^{\phi}{}_{\theta\phi\theta} & R^{\theta}{}_{\theta\theta\phi} + R^{\phi}{}_{\theta\phi\phi} \\ R^{\theta}{}_{\phi\theta\theta} + R^{\phi}{}_{\phi\phi\theta} & R^{\theta}{}_{\phi\theta\phi} + R^{\phi}{}_{\phi\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad (5.103)$$

The Ricci scalar is

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{1}{a^2} + \frac{1}{a^2\sin^2\theta}\sin^2\theta = \frac{2}{a^2}\,. \qquad (5.104)$$

The Gaussian curvature

$$K \equiv \frac{R}{2} = \frac{1}{a^2} \qquad (5.105)$$

is positive and constant, as expected, and coincides with the result (5.42) obtained by directly applying Bertrand-Diquet-Puiseux formula (5.38).

Remember: This was quite an explicit computation to show how to use the symmetries to rapidly derive the final result. In two dimensional cases it is better two remember that the Riemann tensor has only one independent component, directly compute the R_{1212} component

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \qquad \longrightarrow \qquad R_{\theta\phi\theta\phi} = g_{\theta\theta}R^{\theta}{}_{\phi\theta\phi} = a^2\sin^2\theta \qquad (5.106)$$

and contract it with the inverse metric to obtain the scalar of curvature

$$R = g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = \frac{2}{a^2} = \frac{2R_{\theta\phi\theta\phi}}{|g|}\,. \qquad (5.107)$$

Note that the result

$$R = \frac{2R_{\theta\phi\theta\phi}}{|g|} \qquad (5.108)$$

is just a particular version of Eq. (5.91).

Exercise

Compute the intrinsic curvature of the two-dimensional cone in Cartesian and polar coordinates. Interpret the result.