CHAPTER 3

"THE HAPPIEST THOUGHT" OF EINSTEIN'S LIFE

For an observer falling freely from the roof of a house there exists - at least in his immediate surroundings - no gravitational field [. . .]. The observer therefore has the right to interpret his state as 'at rest' (at least until he hits the ground!).

A. Einstein (1920)

The Poisson equation for the gravitational field

$$\nabla^2\Phi(t,x^i) = 4\pi G\rho(t,x^i)$$

is a linear partial differential equation of 2nd order which does not contain any explicit time dependence. The gravitational potential responds *instantaneously* to the changes in the matter distribution! This was awkward even for Newton

> *That one body may act upon another at a distance through a vacuum, without the mediation of anything else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has philosophical matters a competent faculty of thinking, can ever fall into it (Principia, p. 643, Ref. 395).*

and it is in clear contradiction with Special Relativity. The instinctive reaction of many physicist when facing this consistency problem was to apply the recipes used when writing the covariant version of Maxwell equations (promote the operator ∇ to $\Box$, introduce some kind of vector potential A^μ for the gravitational field, generalize the Newtonian force to some combinations of fields and 4-velocities, get retarded potentials, etc . . .). None of the attempts was sucessful[1]. Einstein eventually concluded that a new approach to the problem must be taken. The purpose of this chapter is to present you Einstein's new look on gravity. As you will see, the *new look* turned out to be a real *old look* that went back to Galileo himself.

[1] We will be back to this point in the future.

3.1 Inertial and gravitational masses

Two different masses enter in the Newton's theory of mechanics and gravitation. According to Newton's second law the acceleration $a^i = d^2x^i/dt^2$ experienced by an object is proportional to the exerted force divided by the *inertial mass* of the object

$$f^i = m_I a^i \,, \tag{3.1}$$

independently of the origin of the force. This mass m_I measures the resistance of an object to accelerations. On the other hand, we have the *gravitational mass*, which measures the strength of the gravity (in the same way that the electric charge measures the strength of the electric force). The force exerted on a *gravitational mass* m_G close to the surface of the Earth is given by

$$f^i = m_G \, g^i \,. \tag{3.2}$$

Comparing the expressions (3.1) and (3.2), we conclude that the *acceleration of gravity* should depend *a priori* on the ratio of the *gravitational mass* to the *inertial mass.*

$$a^i = \left(\frac{m_G}{m_I}\right) g^i \,. \tag{3.3}$$

Nevertheless, as verified by Galileo's ramp experiments[2], all bodies fall with the same acceleration in a gravitational field

$$a^i = g^i \,. \tag{3.4}$$

This observation implies the equality of the quantity controlling inertia (m_I) and that measuring the strength of gravity (m_G)

$$m_I = m_G \,, \tag{3.5}$$

for all materials, independently of its composition.

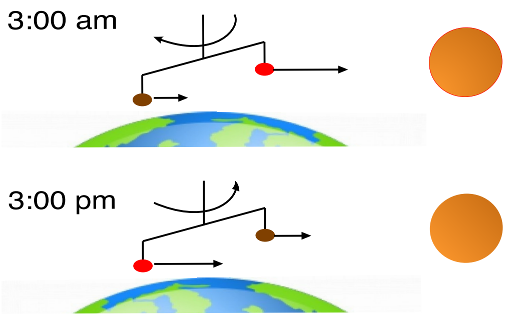
Exercise

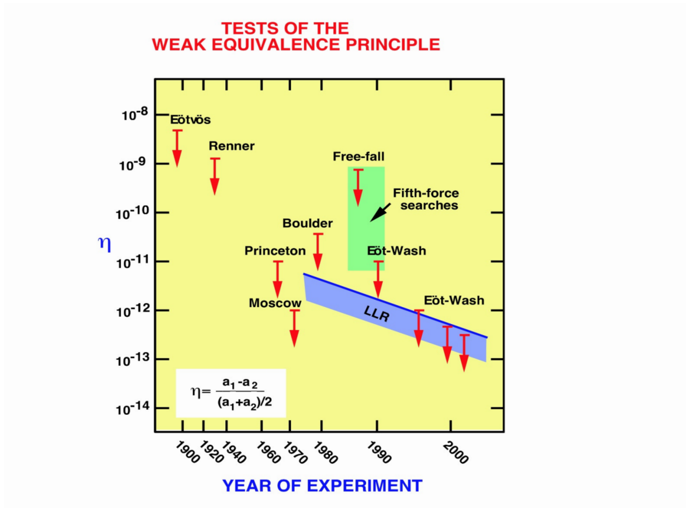
Consider the magnitude of the electrostatic interaction at a distance r between two particles of charges q_1, q_2 and *inertial masses* m_{1i},m_{2i}

$$F_e = \frac{q_1 q_2}{4\pi r^2} \,. \tag{3.6}$$

How does the magnitude of the acceleration felt by particle 2 depends on its properties?

[2] Yes, ramps and a water clock. The image of Galileo dropping balls from the leaning power of Pisa is just a widespread italian legend.





The results of Galileo's experiments were confirmed, among others, by Newton himself and by the Baron Eötvös de Vásárosnamény, who used respectively pendula and a torsion balance with different materials[3].

> **Exercise**
> How does the oscillation period of a simple pendulum depend on the ratio m_I/m_G?

The difference in the acceleration experienced by the two bodies is encoded in the so-called Eötvös parameter

$$\eta = \frac{\Delta a}{a} = \frac{2|a_1 - a_2|}{|a_1 + a_2|} = \sum_I \eta^I \left(\frac{E_1^I}{m_{I,1}c^2} - \frac{E_2^I}{m_{I,2}c^2} \right) , \tag{3.7}$$

where in the last step we have made explicit the contribution of the various energy forms E^I to the difference between inertial and gravitational masses

$$m_G - m_I \equiv \sum_I \eta^I \frac{E^I}{c^2} . \tag{3.8}$$

The experimental results are summarized in the following table.

	$\frac{\vert m_I - m_G \vert}{m_I}$
Rest mass, proton and neutrons	$< 10^{-11}$
Rest mass, electrons	$< 2 \times 10^{-8}$
Electric fields in nucleus	$< 4 \times 10^{-10}$
Magnetic fields in nucleus	$< 2 \times 10^{-7}$
Strong fields in nucleus	$< 5 \times 10^{-10}$
Weak fields in nucleus	$< 10^{-2}$
Gravitational energy in Earth	$< 2 \times 10^{-3}$

[3] Eötvos located two test objects on the opposite ends of a dumbbell suspended from a forsion fiber. If the inertial and gravitational masses of those objects were different the centripetal effects associated with the rotation of the Earth would give rise to a torque (everywhere but at the poles) that could be measured with a delicate torsion balance pointing west-east. For a detailed description of the Eötvös' original experiments see for instance Weinberg's book.

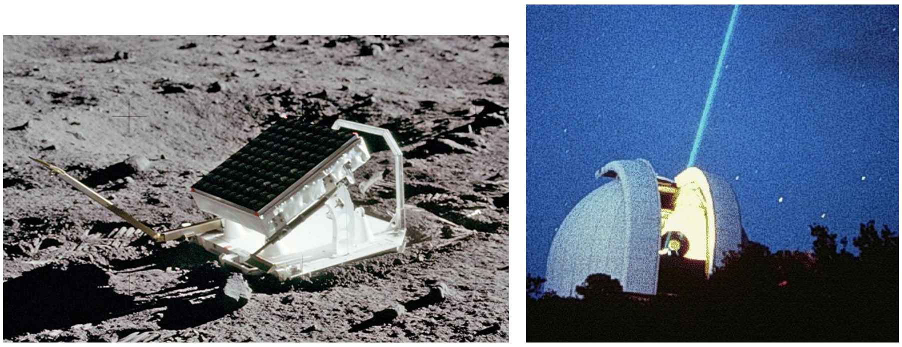


Figure 3.1: Lunar Laser Ranging (LLR) experiments

Note that the displayed cases do not include the contribution of the gravitational self-interaction of the masses, which, for laboratory size experiments, is extremely small. Its contribution can be however tested via the so-called Nördvedt effect. If the gravitational self-energy did not follow Galileo's equivalence principle, the Earth and the Moon would fall at different rates towards the Sun, elongating the orbit of the Moon in the Sun direction. As shown by Lunar Laser Ranging experiments (LLR), which use reflectors that were located in the surface in the Moon during the Apollo 11 mission, cf. Fig. 3.1, the gravitational self-energy behaves as any other energy form, in perfect agreement with Eötvos' results. Indeed, LLR experiments provide the most accurate tests of the equivalence between inertial and gravitational masses. The constrains are really impressive

$$\eta = \frac{2|a_{\rm E} - a_{\rm M}|}{(a_{\rm E} + a_{\rm M})} = (-1 \pm 1.4) \times 10^{-13}\,. \tag{3.9}$$

Not only matter but also antimatter seems to follow the Galileo's result. Important constraints of the order 10^{-9} were obtained by the CPLEAR collaborations from neutral kaon systems[4].

3.2 The Equivalence Principle

The experimental equality of inertial and gravitational masses is a quite surprising and mysterious property, relating two completely different concepts and not required at all for the consistency of Newton's theory. For Galileo and Newton, this was just a coincidence. For Einstein, it would be the first stone in the impressive geometrical edifice of General Relativity. Einstein's theory will be constructed on top of something so simple that even Galileo could have discovered: the relation between gravity and inertia. To illustrate this equality let me consider one of the most famous Einstein's *Gedankenexperimente*. Imagine yourself dropping a ball in the surface of the Earth. You will see the ball falling with constant acceleration. "The effect of gravity", you would say. Now imagine yourself performing the same experiment inside a completely isolated rocket in outer space which moves with a constant acceleration $a = g$. You will observe the same: the ball falling with constant acceleration. Without knowing it, you would not be able to decide if you were in the true gravitational field of the Earth or in a rocket! This apparently trivial observation is summarized in the so-called *Weak Equivalence Principle*:

[4]Under the assumption of exact CPT symmetry.

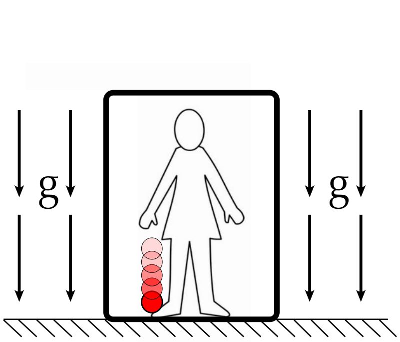


Figure 3.2: Einstein's *Gedankenexperiment*

Einstein's Equivalence Principle

The trajectories of particles in the gravitational field are *locally* indistinguishable from the trajectories of free particles as viewed from an accelerated reference frame.

The gravitational interaction resembles also the pseudo forces resulting from the use of non-inertial reference frames. For example, if there is a frame of reference rotating with angular velocity ω with respect to an inertial reference frame, *all* bodies appear to accelerate spontaneously with the same acceleration in that rotating frame. It seems that there is a universal force acting on all bodies with *a magnitude proportional to their inertial masses*

$$\mathbf{F} = -m_I \left[\dot{\omega} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \right] . \tag{3.10}$$

Accelerated frames and local gravitational forces appear to be intimately related. Both of them act in the same way on all bodies, are proportional to mass and can be transformed away by changing to a suitable reference frame; a local free falling frame in the case of gravity.

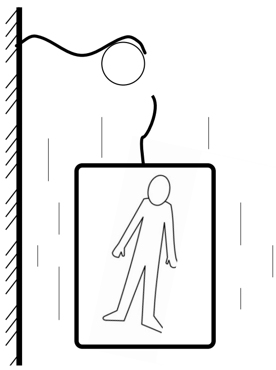


Figure 3.3: A local inertial reference frame.

Exercise

Einstein's toy: A version of the following device was constructed as a birthday present for Albert Einstein. The device consists of hollow broomstick with a cup at the top, together with a metal ball and an elastic string. When the broomstick is held vertical, the ball can rest in the cup. The ball is attached to one end of the elastic string, which passes through a hole in the bottom of the cup, and down the hollow centre of the broomstick to the bottom, where its other end is secured. You hold the broomstick vertical, with your hand at the bottom, the cup at the top, and with the ball out of the cup, suspended on its elastic string. The tension in the string is not enough to draw the ball back into the cup. The problem is to find an elegant way to get the ball back into the cup. (Inelegant ways are: using your hands or shaking the stick up and down).

3.3 Life in the rocket: Rindler spacetime

Since accelerated reference frames mimic the local effects of the gravitational field, the understanding of their properties seems to be a first step towards the correct description of gravity. Let me start by analyzing the rocket *Gedankenexperiment* presented above.

Note the slight change of notation below. From now on, we will reserve the first letters of the Greek alphabet $\alpha, \beta, \ldots$ for indices associated to *inertial* reference frames. Intermediate letters of the Greek alphabet $\mu, \nu, \ldots$ will stand for general (non-inertial, accelerated) reference frames.

Consider the movement of the accelerated rocket from the point of view of an inertial observer $\{\xi^\alpha\}$, *momentarily* at rest with respect to the rocket's trajectory. The orientation of the coordinate grid is such that the rocket moves along the ξ^3-direction. In that *instantaneous* inertial frame, the rocket is seen to undergo a constant acceleration g

$$u^\mu = (1,0,0,0)\,, \qquad a^\mu u_\mu = 0 \quad \rightarrow \quad a^\mu = (0,0,0,g)\,, \qquad a^\mu a_\mu = g^2\,. \tag{3.11}$$

In order to determine the wordline $\xi^\alpha(\tau)$ of the rocket at later times, let us look for a general solution of the covariant equation $u^\alpha u_\alpha = -1$. Writing it explicitly

$$\eta_{\alpha\beta} u^\alpha u^\beta = -(u^0)^2 + (u^3)^2 = -1\,, \tag{3.12}$$

the solution becomes pretty obvious

$$u^0 = \cosh f(\tau)\,, \qquad u^3 = \sinh f(\tau)\,. \tag{3.13}$$

The unknown function $f(\tau)$ can be determined by taking the derivative of the last two equations

$$a^\alpha = \frac{du^\alpha}{d\tau} = \dot{f}(\tau)\,(\sinh f(\tau), 0, 0, \cosh f(\tau)) \tag{3.14}$$

and imposing the covariant condition $a^\mu a_\mu = g^2$. We get $g^2 = \dot{f}^2$, $f(\tau) = g\tau$ and

$$u^\alpha = (\cosh g\tau, 0, 0, \sinh g\tau)\,. \tag{3.15}$$

The work is almost done. Integrating (3.15) with the initial condition $\xi^\alpha(0) = (0,0,0,g^{-1})$, we get

$$\xi^\alpha(\tau) = g^{-1}(\sinh g\tau, 0, 0, \cosh g\tau)\,. \tag{3.16}$$

The "constantly accelerated" rocket describes an equilateral hyperbola with semi-major axis $1/g$

$$(\xi^3)^2 - (\xi^0)^2 = g^{-2}\,. \tag{3.17}$$

Let us now look at the problem from the point of view of an accelerated observer sitting in the rocket. Since the transformation from inertial to accelerated frames is not a Lorentz transformation, we should expect a change in the Minkowski line element ds^2. The natural coordinates for the accelerated observer are those adapted to its trajectory. Let's call them $(x^0, x^3) = (\eta, \rho)$. The transformation to this frame takes the form[5]

$$\xi^0(\eta,\rho) = \rho \sinh\eta\,, \qquad \xi^3(\eta,\rho) = \rho\cosh\eta\,. \tag{3.18}$$

In terms of the new coordinates η and ρ, the Minkowski line element $ds^2 = -(d\xi^0)^2 + (d\xi^3)^2$ becomes modified

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2 \equiv g_{\mu\nu}dx^\mu dx^\nu\,. \tag{3.19}$$

The metric $g_{\mu\nu} = \mathrm{diag}(-\rho^2, 1)$ is now space-time dependent!

3.4 Beyond inertial observers

Let us formalize the concepts appearing in the previous example. The distance between two neighboring points, as measured in a local inertial frame at rest with respect to the particle[6], is given by the Minkowski metric $\eta_{\alpha\beta}$

$$ds^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta\,. \tag{3.20}$$

When performing a transformation to a general coordinate system[7] $x^\mu = x^\mu(\xi^\alpha)$, the line element becomes modified

$$ds^2 = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}dx^\mu dx^\nu = g_{\mu\nu}dx^\mu dx^\nu\,. \tag{3.21}$$

and distances are no longer determined by the Minkowski metric, but rather by a metric

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}\,, \tag{3.22}$$

which generically depends on the spacetime coordinates. The inverse of the new metric is defined through the relation $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu{}_\lambda$ and can be easily computed by taking into account the identity

$$\frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial\xi^\beta}{\partial x^\mu} = \delta^\beta{}_\alpha\,. \tag{3.23}$$

We get

$$g^{\mu\nu} = \eta^{\alpha\beta}\frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial x^\nu}{\partial\xi^\beta}\,. \tag{3.24}$$

Exercise

Prove the *similarity transformation* (3.24).

[5]The change of coordinates is just the Lorentzian analogue of polar coordinates. inspired on the wordline equation (3.16).

[6]In the context of a particle in the gravitational field these frames are called *local free falling reference frames*.

[7]General means completely arbitrary. It can be a curvilinear coordinate system, an accelerated system, a rotating system . . . whatever you want.

Note that the reference frame x^μ is not at all privileged. We could perfectly move now into another non-inertial reference frame $\bar{x}^\rho(\xi^\alpha)$ in which

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\rho}d\bar{x}^\rho \frac{\partial x^\nu}{\partial \bar{x}^\sigma}d\bar{x}^\sigma = g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\rho}\frac{\partial x^\nu}{\partial \bar{x}^\sigma}d\bar{x}^\rho d\bar{x}^\sigma = \bar{g}_{\rho\sigma}d\bar{x}^\rho d\bar{x}^\sigma \,, \tag{3.25}$$

with

$$\bar{g}_{\rho\sigma} = g_{\mu\nu}\frac{\partial x^\mu}{\partial \bar{x}^\rho}\frac{\partial x^\nu}{\partial \bar{x}^\sigma} \,. \tag{3.26}$$

The results of this section are summarized in Figure 3.4.

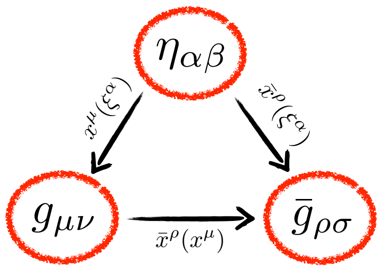


Figure 3.4

Exercise
Show that $g_{\mu\nu}$ must be symmetric, i.e. $g_{\mu\nu} = g_{\nu\mu}$.

3.5 The geodesic equation

The equation of motion for a free particle in an accelerated reference frame can be obtained by applying the transformation (3.22) to the Lagrangian of a free relativistic particle (2.51). We get

$$S = \frac{1}{2}\int d\sigma \left(e^{-1}(\sigma)g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - m^2 e(\sigma)\right) \,. \tag{3.27}$$

As in the Minkowski case, the action (3.27) is invariant under reparametrizations of the path $\sigma \to \sigma = f(\sigma)$ provided that we let $e(\sigma)$ transform in such a way that the quantity $e(\sigma)d\sigma$ is left invariant. Note also that the action (3.27) is invariant under general coordinate transformations[8], as can be easily seen by taking into account the similarity relation (3.26).

The Euler-Lagrange equation $\frac{\partial L}{\partial e} = 0$ for the non-dynamical variable $e(\sigma)$

$$g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = -m^2 e(\sigma)^2 \,, \tag{3.28}$$

automatically incorporates the massive ($e(\sigma) = 1/m$) and massless ($e(\sigma) = 1, m \to 0$) cases we are interested in. In these two limits, the action (3.27) takes the form

$$S_{\text{massive}} = \frac{1}{2}m\int d\sigma \left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} - 1\right) \,, \qquad S_{\text{massless}} = \frac{1}{2}\int d\sigma \left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right) \,, \tag{3.29}$$

[8]This should be expected from pure Lagrangian mechanics.

with $\sigma = \tau$ for the massive case. S_{massive} and S_{massless} are very similar. Indeed, the computation of the equations of motion is formally equivalent in both cases[9]. Let us denote by a dot the derivative with respect to σ and forget in what follows about the irrelevant factors m and $m/2$. The equations of motion

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\rho}\right) - \frac{\partial L}{\partial x^\rho} = 0 \tag{3.30}$$

for the generalized coordinates x^ρ can be computed as follows. The simplest part is the variation of $1/2 g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ with respect to x^ρ. All the dependence on the coordinates is hidden in the metric

$$\frac{\partial L}{\partial x^\rho} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu \dot{x}^\nu \,. \tag{3.31}$$

The variation with respect to $\dot{x}^\rho$ is slightly more involved, but can be however written in a very compact way by taking into account the properties

$$\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\rho} = \delta^\mu{}_\rho \qquad g_{\mu\nu}\delta^\nu{}_\rho = g_{\mu\rho}\,, \qquad g_{\mu\nu} = g_{\nu\mu}\,, \tag{3.32}$$

together with some simple index relabeling. We get

$$\frac{\partial}{\partial \dot{x}^\rho}\left(\frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu\right) = \frac{1}{2}\left(g_{\mu\nu}\frac{\partial \dot{x}^\mu}{\partial \dot{x}^\rho}\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu\frac{\partial \dot{x}^\nu}{\partial \dot{x}^\rho}\right) = \frac{1}{2}\left(g_{\rho\nu}\dot{x}^\nu + g_{\mu\rho}\dot{x}^\mu\right) = g_{\rho\nu}\dot{x}^\nu \,. \tag{3.33}$$

Collecting the two pieces, the Euler-Lagrange equations (3.30) become

$$\begin{aligned} \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}^\rho} - \frac{\partial L}{\partial x^\rho} &= \frac{d}{d\sigma}\left(g_{\rho\nu}\dot{x}^\nu\right) - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu\dot{x}^\nu \\ &= \frac{\partial g_{\rho\nu}}{\partial x^\sigma}\dot{x}^\sigma\dot{x}^\nu + g_{\rho\nu}\ddot{x}^\nu - \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial x^\rho}\dot{x}^\mu\dot{x}^\nu \\ &= g_{\rho\nu}\ddot{x}^\nu + \dot{x}^\nu\dot{x}^\sigma\left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} - \frac{1}{2}\frac{\partial g_{\sigma\nu}}{\partial x^\rho}\right) \\ &= g_{\rho\nu}\ddot{x}^\nu + \frac{1}{2}\,\dot{x}^\nu\dot{x}^\sigma\left(\frac{\partial g_{\rho\nu}}{\partial x^\sigma} + \frac{\partial g_{\rho\sigma}}{\partial x^\nu} - \frac{\partial g_{\sigma\nu}}{\partial x^\rho}\right) \\ &= 0\,. \end{aligned} \tag{3.34}$$

The work is done. Multiplying by the inverse metric and relabeling indices we obtain the equation we were looking for, the so-called *geodesic equation*

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0\,, \qquad \Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\rho g_{\sigma\nu} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}\right)\,. \tag{3.35}$$

Exercise

- Consider a reparametrization $\sigma \to f(\sigma)$. Show that the geodesic equation (3.35) retains its form only if $f(\sigma) = a\sigma + b$.
- Compute the geodesic equation associated to the Rindler metric (3.16).

The geodesic equation is automatically covariant since the Lagrangian from which it was derived was invariant under general coordinate transformations. The transformation of the so-called *Christoffel symbols* $\Gamma^\mu{}_{\nu\lambda}$ is however non-homogeneous

$$\bar{\Gamma}^{\mu'}{}_{\nu'\rho'} = \Gamma^\mu{}_{\nu\rho}\frac{\partial \bar{x}^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial \bar{x}^{\nu'}}\frac{\partial x^\rho}{\partial \bar{x}^{\rho'}} + \frac{\partial \bar{x}^{\mu'}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial \bar{x}^{\nu'}\partial \bar{x}^{\rho'}}\,. \tag{3.36}$$

[9] The only difference is the presence of a global factor m and a constant term $m/2$ which do not play any role in the variation of the action

They are *not* a tensor.

Exercise

- Which is the form taken by Eq. (3.37) in a Cartesian coordinate system?
- Prove the transformation law (3.36) using the fact the the geodesic equation is covariant.
- How many independent components have the Christoffel symbols in four dimensions?

The Christoffel symbols encode the *local* aspects of the gravitational interaction as well as the *fictitious* forces (centrifugal, Coriolis, etc)

$$F^\mu \equiv \frac{d^2x^\mu}{d\sigma^2} = -\Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma} \tag{3.37}$$

arising when using non-inertial reference frames. This kind of forces can always be eliminated by going to an inertial reference frame or to a *free-falling frame*. Note that this would not be the case if the Christoffel symbols were tensors.

Local free-falling reference frames

The geodesic equation allows for a precise definition of *local free-falling frames*. According to Equivalence Principle, in those frames the geodesic equation must become

$$\left.\frac{d^2\xi^\alpha}{d\sigma^2}\right|_P = 0\,. \tag{3.38}$$

A free-falling reference frame at P is therefore defined as[a]

$$g_{\mu\nu}(P) = \eta_{\mu\nu}\,, \qquad \partial_\sigma g_{\mu\nu}(P) = 0 \tag{3.39}$$

with the second condition being equivalent to the vanishing of the Christoffel symbols at that point, i.e. $\Gamma^\mu{}_{\nu\lambda}(P) = 0$.

[a] The existence of these frames is guaranteed by the so-called *Local flatness theorem*.

3.5.1 Massive particles don't go on diet

It is easy to show that the geodesic equation (3.35) is equivalent to a conservation equation

$$\frac{d^2x^\mu}{d\sigma^2} + \Gamma^\mu{}_{\nu\rho}\frac{dx^\nu}{d\sigma}\frac{dx^\rho}{d\sigma} = 0 \qquad \rightarrow \qquad \frac{d}{d\sigma}\left(g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\right) = 0 \tag{3.40}$$

associated to translations in the parameter σ, $f(\sigma) = \sigma + c$. The physical meaning of Eq. (3.40) is quite obvious and should be expected; it simply states that massless/massive particles remain massless/massive along the geodesic. In other words, given an initial condition

$$\left. g_{\mu\nu}\frac{dx^\nu}{d\sigma}\frac{dx^\lambda}{d\sigma}\right|_{\sigma=0} = \begin{cases} -1 & \text{for massive particles} \\ \;\;0 & \text{for massless particles} \end{cases}\,, \tag{3.41}$$

it will by satisfied for all values of σ[10]. We will rediscover this equation in Chapter 5, when dealing with the concept of parallel transport.

[10] Note that for the massive case we can identify $\sigma = \tau$

Exercise
Prove the relation (3.40).

3.5.2 Conserved quantities

Note that if the metric coefficients are independent of one coordinate[11] x^ν, the Lagrangian (3.27) will be also independent of such a coordinate. In such a case, the covariant component $\dot{x}_\nu$ is a conserved quantity along affinity parametrized geodesics

$$p_\nu = \frac{\partial L}{\partial \dot{x}^\nu} = g_{\mu\nu}\dot{x}^\mu = \dot{x}_\nu = \text{constant}\,. \tag{3.42}$$

We will make use of this important property in Chapter 9, when dealing with the Schwarzschild geometry.

3.6 The Newtonian limit

Let us see how the usual results of Newtonian gravity fit into the geometric picture. Of course, we cannot expect to link the relativistic formulation presented above with a non-relativistic theory of gravity without doing some assumptions. We will consider a massive particle moving at small velocity (with respect to the speed of light)

$$\frac{dx^i}{dt} \ll 1 \qquad \longrightarrow \qquad \frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \tag{3.43}$$

in a "weak"

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll 1 \tag{3.44}$$

and stationary[12] gravitational field

$$\partial_0 g_{\mu\nu} = \partial_0 h_{\mu\nu} = 0\,. \tag{3.45}$$

The first two conditions (small velocities and weak fields) are quite natural from the point of view of a non-relativistic description. On the other hand, the stationarity condition (3.45) is just a good approximation for the particular cases we will be interested in in this section: the gravitational fields of the Sun and the Earth.

At first order in the small perturbation $h_{\mu\nu}$, the geodesic equation (3.35) takes the form

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\,, \tag{3.46}$$

where the Christoffel symbols Γ^μ_{00} are completely determined by the perturbation $h_{\mu\nu}$ around the Minkowsky metric[13]

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{0\rho}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^\rho}\right) = -\frac{1}{2}g^{\mu\rho}\frac{\partial g_{00}}{\partial x^\rho} = -\frac{1}{2}\eta^{\mu\rho}\frac{\partial h_{00}}{\partial x^\rho}\,. \tag{3.47}$$

[11] The coordinate x^ν is then said to be *cyclic*.
[12] Or varying sufficiently slow over the scale probed by the particle.
[13] Note that, since we are interested only in first order terms, we can raise and lower indices with the Minkowski metric $\eta_{\mu\nu}$. For example $h^\mu{}_\nu = g^{\mu\lambda}h_{\lambda\nu} \simeq \eta^{\mu\lambda}h_{\lambda\nu} + \mathcal{O}(h^2_{\mu\nu})$.

	Mass	Size	$\|\Phi\|/c^2$
Atome	10^{-26} Kg	10^{-10} m	10^{-43}
Human	10^{2} Kg	1 m	10^{-25}
Earth	10^{25} Kg	10^{7} m	10^{-9}
Sun	10^{30} Kg	10^{9} m	10^{-6}
Galaxy	10^{41} Kg	10^{21} m	10^{-7}
White Dwarf	10^{30} Kg	10^{7} m	10^{-4}
Neutron Star	10^{30} Kg	10^{4} m	0.1
Black Hole			1

Table 3.1: Gravitational self-energy: Orders of magnitude

Splitting the spatial and temporal components of Eq. (3.46) and using the stationarity condition (3.45), we obtain[14]

$$\frac{d^2t}{d\tau^2} = 0\,, \qquad \frac{d^2x^i}{d\tau^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}\,, \tag{3.48}$$

The first of these two equations allows us to identify the proper time τ with the coordinate time t and write

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\frac{\partial h_{00}}{\partial x^i}\,. \tag{3.49}$$

The value of the unknown function h_{00} can be determined by comparing Eq. (3.48) with the Newtonian expression for a particle in a gravitational field

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\frac{\partial \Phi}{\partial x^j}\,. \tag{3.50}$$

The first *true* component of the gravitational metric tensor[15] comes directly from Newton's theory!

$$h_{00} = -\frac{2\Phi}{c^2} \qquad \longrightarrow \qquad g_{00} = -\left(1+\frac{2\Phi}{c^2}\right)\,. \tag{3.51}$$

Indeed…this is the *first* and the *last* component that we can expect to get from Newton … Newtonian gravity involves just one scalar function: the gravitational potential Φ, nothing else. This observation naturally raises the question of how to compute the remaining components of the metric. Let's forget about this problem for a while and enjoy our findings. As you will see, we can learn a lot of new things without knowing the precise form of the other metric components. The correction to the Minkowski metric is proportional to the so-called *gravitational self-energy* Φ/c^2. This quantity can be understood as the ratio of the Newtonian potential energy to the relativistic energy. For an object of mass M and typical size R we have

$$\frac{|\Phi|}{c^2} = \frac{GM^2}{R}\cdot\frac{1}{Mc^2} = \frac{GM}{Rc^2}\,. \tag{3.52}$$

[14] Note that we have restored the speed of light c for later convenience.

[15] Note that we could in principle allow for an extra constant C in Eq. (3.51), i.e $h_{00} = -\frac{2\Phi}{c^2} + C$, which should be fixed by requiring the metric to approach the flat Minkowski metric at infinity. For isolated mass distributions the gravitational potential Φ vanishes at infinity and therefore $C = 0$.

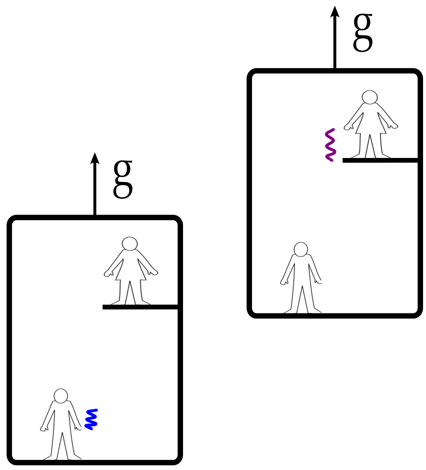


Figure 3.5: Gravitational redshift, *Gedankenexperiment*.

Some orders of magnitude for Φ can be found in Table 3.1. Note that, even for a white dwarf or a galaxy, the gravitational self-energy is very small; the weak field approximation used in the derivation of Eq. (3.51) is justified. The correction to the Minkowski metric is expected to be important only for very compact object such as a neutron star or a black hole.

3.7 The power of the equivalence principle

Let us consider the direct consequences of the previous results. In order to get some intuition let me go back for a moment to the constantly accelerated rocket and perform the following Gedankexperiment. Imagine two observers in the rocket, one on the base and one on a platform close to the ceiling. The observer at the bottom sends some light pulses with a frequency dictated by its proper time interval $\nu_1 = 1/\Delta\tau_1$. Due to the acceleration of the rocket, these pulses are received by the observer at the top at a lower rate $\nu_2 = 1/\Delta\tau_2$ than the rate at which they were emitted[16]. According to the Equivalence Principle the same phenomenon should happen in the presence of gravity. Yes, gravity must affect the flow of time!

Let us put our Gedankenexperiment into equations. Having a look to Eq. (3.51), we see that the interval in proper time $d\tau$ at a fixed point in the vicinity of a massive object differs from the interval in coordinate time dt

$$d\tau = \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \sqrt{1 + \frac{2\Phi(r)}{c^2}}dt\,. \tag{3.53}$$

A local measurement of this effect is nevertheless impossible since our measure instruments are affected by gravity in the same way that the timing of the objects we want to measure. Observable effects on the flow of time can only appear when we compare two different points in the gravitational potential Φ, as we did in the rocket *Gedankenexperiment*.

3.7.1 Gravity and the flow of time

Consider two observers in the weak gravitational field of a spherically symmetric and stationary mass distribution. Although the Newtonian limit developed in the Section 3.6 provides only information

[16] By the time at which the light arrives to the top the ceiling is moving faster than when the light was emitted.

about the g_{00} element, the large symmetry of the problem severely constrains the form of the metric to be

$$ds^2 = -\left(1 + \frac{2\Phi(r)}{c^2}\right) dt^2 + g_{rr}(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,, \tag{3.54}$$

with $g_{rr}(r)$ an undetermined function of the radial distance, whose explicit form will not be needed in what follows. In order to disentangle the effect of gravity from other velocity dependent Doppler-like effects and to make the analysis as clear as possible, we will require the observers to be at rest in a radial configuration with coordinates r_1 and r_2. Imagine the observer at r_1 sending pulses of light to the observer at r_2. The period of emitted pulses is the interval in proper time of the emitter

$$\Delta\tau_1 = \int \sqrt{-g_{00}(r_1)}dt = \sqrt{-g_{00}(r_1)} \int dt = \sqrt{-g_{00}(r_1)}\Delta t_1 \,. \tag{3.55}$$

On the other hand, the period of received pulses is the interval in proper time of the receiver

$$\Delta\tau_2 = \int \sqrt{-g_{00}(r_2)}dt = \sqrt{-g_{00}(r_2)} \int dt = \sqrt{-g_{00}(r_2)}\Delta t_2 \,. \tag{3.56}$$

The coordinate interval elapsed between the emission of two pulses Δt_1 is equal to the coordinate time interval elapsed between the reception on two pulses Δt_2, as can be easily seen by noting that the coordinate time interval needed to go from r_1 to r_2

$$ds^2 = -g_{00}dt^2 + g_{rr}dr^2 = 0 \qquad \longrightarrow \qquad \Delta t = \int_{r_1}^{r_2} dr \left(\frac{-g_{rr}(r)}{g_{00}(r)}\right) \tag{3.57}$$

is independent of the coordinate time t. Taking the ratio of Eqs. (3.56) and (3.55), we get the first prediction of the Equivalence Principle

$$\frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 + 2\Phi(r_2)/c^2}{1 + 2\Phi(r_1)/c^2}} \,. \tag{3.58}$$

For weak gravitational fields, the previous expression can be approximated by its binomial expansion[17]

$$\frac{\Delta\tau_2}{\Delta\tau_1} \simeq 1 + \frac{\Phi(r_2) - \Phi(r_1)}{c^2} \,, \tag{3.59}$$

which is usually quoted in terms of the ratio

$$\frac{\Delta\tau}{\tau} \equiv \frac{\Delta\tau_2 - \Delta\tau_1}{\Delta\tau_1} = \frac{\Phi(r_2) - \Phi(r_1)}{c^2} \,. \tag{3.60}$$

> **Time goes by slower**
>
> Clocks slow down in those places where the gravitational potential is larger (in magnitude). In particular, clocks at a distance r from the surface of a massive spherical body of mass M ($\Phi = -\frac{GM}{r}$) slow down by a factor $\sqrt{1 - \frac{2GM}{rc^2}}$ with respect to clocks at $r \to \infty$ ($\Phi = 0$).

The dilation of time was tested by Hafele and Keating in 1972 using cesium-beam atomic clocks transported on commercial flights around the Earth and compared on return to standard clocks in the US Naval Observatory. The net effect on the reading of the on-flight clocks is a combination of special relativistic effects and gravitational changes in the flow of time. The two contributions act in

[17] $(1+x)^{1/2} = 1 + \frac{1}{2}x$.

Time of flight	41.2 h	48.6 h
$\Delta\tau$ (ns)	Eastward	Westward
$\Delta\tau_{\mathrm{G}}$	144 ± 14	179 ± 18
$\Delta\tau_{\mathrm{SR}}$	-184 ± 18	96 ± 10
$\Delta\tau_{\mathrm{tot}}$	-40 ± 23	275 ± 21
$\Delta\tau_{\mathrm{obs}}$	-59 ± 10	273 ± 7

Table 3.2: Hafele Keating: Predictions and experimental results

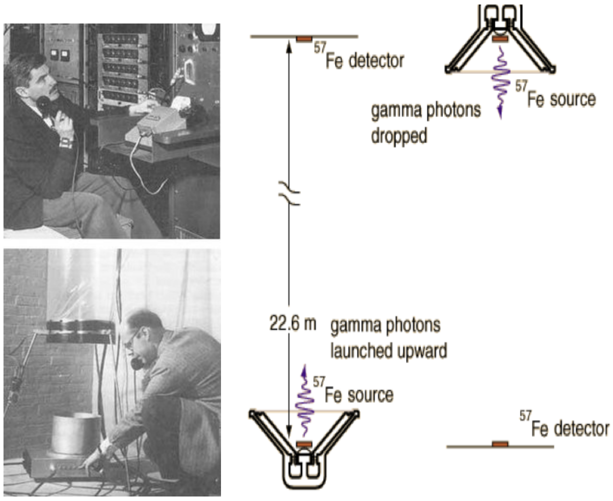


Figure 3.6: The highs and lows: Redka and Pound at the top and bottom of the tower.

an opposite way. Special Relativity tends to decrease the rate of the clock in the plane with respect to the standard clock in the surface of the Earth[18]. On the other hand, gravity tends to speed up the clock in the plane with respect to the clock in the stronger gravitational field of the Earth. The experiment was performed twice, once flying towards the east and once flying towards the west. The results and their comparison with the predictions are summarized in Table 3.2. As you can see, the agreement between the theory and the theoretical prediction is notably good.

Exercise

How older are the theorists of the upper floor of the Cubotron with respect to the experimentalists in the lower floor at the end of their academic life? Should this effect be taken into account by the Swiss pension system?

[18]This is just a consequence of the well-known time dilation effect in Special Relativity.

3.7.2 Gravitational shift of frequencies

An immediate consequence of the previous result is the gravitational frequency shift. Denoting by ν_1 the frequency of the light emitted at r_1 and by ν_2 the frequency of the light received at r_2, Eq. (3.58) can be rewritten as

$$\frac{\nu_1}{\nu_2} \equiv 1 + z = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} = \sqrt{\frac{1 + 2\Phi(r_2)/c^2}{1 + 2\Phi(r_1)/c^2}}\,, \tag{3.61}$$

where we have defined the so-called *redshift parameter*

$$z = \frac{\Delta\nu}{\nu} \equiv \frac{\nu_1 - \nu_2}{\nu_2}\,. \tag{3.62}$$

It $z > 0$ the received light is said to be *redshifted*, while if $z < 0$ the light is said to be *blueshifted*. For weak gravitational potentials, Eq. (3.61) can be approximated by

$$z = \frac{\Delta\nu}{\nu} \simeq \frac{\Phi(r_2) - \Phi(r_1)}{c^2}\,. \tag{3.63}$$

To get an estimate of the order of magnitude of this frequency shift, consider for instance the light from the Sun ($r = r_1$) received on Earth ($r = r_2$)[19]. Since $r_2 > r_1$, $|\Phi(r_1)| > |\Phi(r_2)|$ and therefore[20] $z > 0$. As in our *Gedankenexperiment*, the light redshifts ($\nu_1 > \nu_2$) as it climbs upwards in the gravitational potential!

The gravitational frequency shift is a test of the Equivalence Principle, not of the Einstein's theory of gravity in its full form. Note that the spatial part of the metric $g_{rr}(r)$ did not played any role in the previous developments.

Numerically, the gravitational redshift of the light emitted by the Sun is very small

$$\frac{\Delta\nu}{\nu} = 2.12 \times 10^{-6}\,, \tag{3.64}$$

and indeed very difficult to detect due to the broadening of spectral lines and to Doppler shifts associated to the convection currents in the solar atmosphere[21].

A more precise non-astronomical test of the gravitational frequency shift was performed by Pound and Redka in 1960 using gamma rays produced in a 14.4 keV atomic transition in ^{57}Fe. These gamma rays were emitted at the top of a tower of 22.6 meters in the Jefferson Physical Laboratory at Harvard university and directed down towards a similar sample of ^{57}Fe located at the bottom of the tower. The absorption of the gamma rays by the receiver is only efficient if the frequency at reception coincides with the frequency at emission (Mössbauer effect). Due to the gravitational shift of frequencies this was not the case. Pound and Rebka compensated the gravitational shift in a very clever way: a Doppler shift induced by the vertical motion of the source at the top of the tower. By looking for a resonance in the absorption they were able to obtain a direct measurement of the gravitational redshift. The result was in excellent agreement with the Equivalence Principle's prediction

$$\frac{(\Delta\nu/\nu)_{\rm exp}}{(\Delta\nu/\nu)_{\rm th}} = 1.05 \pm 0.10\,. \tag{3.65}$$

[19] $\Phi(r_1) = -GM_\odot/r_1$ and $\Phi(r_2) = -GM_\odot/r_2$ are small (cf. Table 3.1). The binomial expansion is justified. The gravitational field of the Earth is neglected.

[20] Remember that the gravitational potential is negative.

[21] The gravitational redshift (3.64) corresponds numerically to the Doppler shift associated to a velocity of 0.6 Km/h, which is easily exceed by the hot gases in the surface of the Sun.

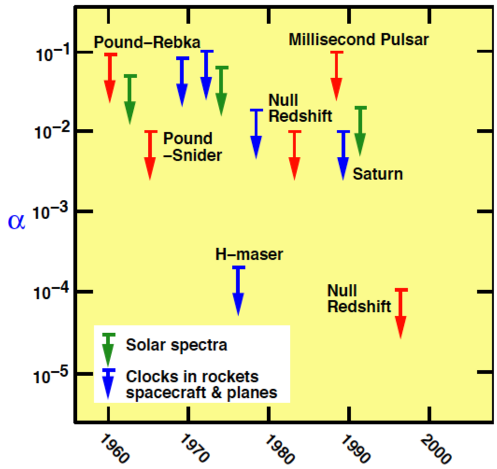


Figure 3.7: Different tests of the gravitational redshift. The parameter α parametrizes the deviations from the Equivalence Principle, $\Delta\nu/\nu = (1+\alpha)\Delta\Phi/c^2$.

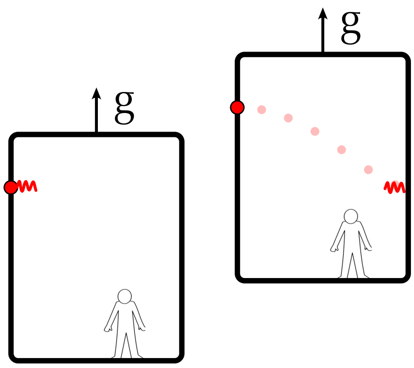


Figure 3.8: Deflection of light, *Gedankenexperiment*.

Exercise
Determine the predicted value $\Delta\nu/\nu$ in the Pound-Rebka experiment. Are the gamma rays traveling down the tower blueshifted or redshifted?

3.8 The weakness of the Equivalence Principle

A second (and incomplete) prediction of the Equivalence Principle is the deflection of light. This effect can be easily understood with another *Gedankenexperiment*. Imagine an observer in the accelerated rocket. A pulse of light is emitted by some device from one of the walls in the transverse direction to the rocket motion. Due to the acceleration of the rocket, the pulse will hit the opposite wall at a height below that of the emission. Since the uniform acceleration of the rocket is locally indistinguishable from a gravitational field, we should expect the same *deflection of light* in a gravitational field.

The bending of light in a gravitational field was considered by Newton himself, but he didn't performed any proper computation. The first known result about the deflection of light was presented by the German astronomer Johann Georg von Soldner in 1804. Based on Newton's corpuscular theory of light, Soldner predicted a deflection angle of $0.87''$ for a ray of light grazing the surface of the Sun. Einstein, unaware of Soldner's computations and based on the Equivalence Principle, obtained the same number one hundred years later, in 1911. Let us reproduce his arguments and (wrong) results.

3.8.1 Einstein's 1911 (wrong) treatment

In Einstein's 1911 paper, the speed of light is considered as a *scalar* quantity which depends on the value of the gravitational field

$$ds^2 = -(1+2\Phi)dt^2 + dX^2 = 0 \qquad \rightarrow \qquad c = c_0\,(1+\Phi) \tag{3.66}$$

The Minkowski value $c = c_0$ is only recovered at long distances ($r \to \infty$), where the gravitational potential is negligible[22]. According to Huygens' principle the position of a wavefront at a time $t + \Delta t$ can be determined by considering each point of the wavefront at t as a source of spherical waves. The wavefront at $t + \Delta t$ is then given by the envelope of the multiple spherical wavefronts originated at t. Imagine a wave front in the vicinity of a matter distribution M. Consider two points P_1 and P_2 separated by a spatial distance δl at time t. The velocity of light at those points (c_1 and c_2) depends of the value of the gravitational field. Having a look to Fig. **??**, we conclude that in a time δt the wavefront in refracted by an angle

$$\delta\alpha = \frac{(c_1 - c_2)\,\delta t}{\delta l} = \frac{\delta\Phi}{\delta l}\delta t\,, \tag{3.67}$$

with $\delta\Phi/\delta l$ the component of the gravitational acceleration along the wavefront. This infinitesimal refraction angle can be integrated along the full path to obtain the total deflection angle[23]

$$\alpha = \int d\alpha = \int \frac{d\Phi}{dl} dt\,. \tag{3.68}$$

Since the velocity of light along the path is nearly constant we can set $dt = ds$, with s measuring the distance along the path. Evaluating the integral (3.68) for an impact parameter b, we get

$$\alpha = \int \frac{d\Phi}{dl} ds = \int_{-\pi/2}^{\pi/2} \frac{GM}{r^2}\cos\theta ds = \frac{2GM}{b}\,, \tag{3.69}$$

which for the particular case of a photon grazing the surface of the Sun becomes[24]

$$\alpha = \frac{2GM_\odot}{c^2 R_\odot} \approx 0.875''\,. \tag{3.70}$$

[22] In "Relativity,The Special and General Theory", Einstein wrote:

> *[...] our result shows that, according to the general theory of relativity, the law of the constancy of the velocity of light in vacuo, which constitutes one of the two fundamental assumptions in the special theory of relativity and to which we have already frequently referred, cannot claim any unlimited validity. A curvature of rays of light can only take place when the velocity of propagation of light varies with position. Now we might think that as a consequence. of this, the special theory of relativity and with it the whole theory of relativity would be laid in the dust. But in reality this is not the case. We can only conclude that the special theory of relativity cannot claim an unlimited domain of validity; its result hold only so long as we are able to disregard the influences of gravitational fields on the phenomena (e.g. of light).*

[23] This is a good approximation for small deflections angles, as is the case of the deflection of light by the Sun.

[24] Note that we have restored the powers of c.

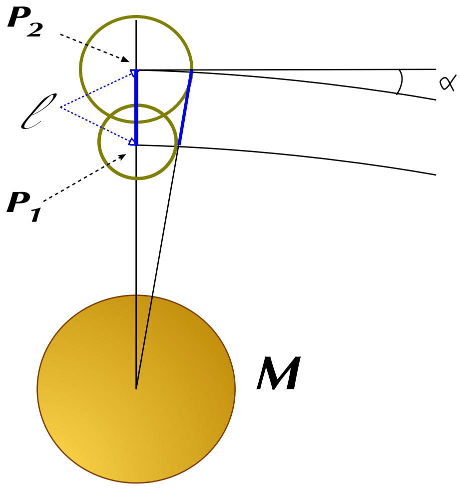


Figure 3.9: Deflection of light, Huygens's principle.

As Einstein stated in the original paper, since 'the fixed stars in the part of the sky near the sun are visible during a total eclipse of the sun, this consequence of the theory may be compared to experiment". He indeed "urgently wishers astronomers to take up this question" and measure the deflection of light during a solar eclipse. Fortunately for him ... they didn't do it on time. Einstein's 1911 prediction based only in the equivalence principle was incomplete[25]. No measurement of the deflection angle was performed between 1911 and 1915, the moment at which he straightens out his result to

$$\alpha = 2 \times \frac{2GM_\odot}{c^2 R_\odot} \approx 2 \times 0.875'' \,. \tag{3.71}$$

Although different expeditions to observe solar eclipses were organized, all of them were cancelled, either for climatological or political reasons. One of the most interesting stories is that of the German astronomer and mathematician Erwin Finlay Freundlich, which, interested on testing Einstein's prediction, convinced the german armament manufacturer Krupp to finance a trip to Crimea on 21st August 1914. Unfortunately for him, and fortunately for Einstein, the German astronomer was arrested by the Russians as a suspected spy before being able to perform any measurement.

[25] We will see why later on.